Ariel University

The Faculty of Electric and Electronic Engineering

Extended Lyapunov Analysis and Simulative Investigations in Stability of Proportional Navigation Guidance Systems

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List of Symbols and Abbreviations

a	Acceleration
a_c	Acceleration command
a_n	Polynomial variable of n^{th} order
α_j	$j = 1, 2$ extremum functions of Lyapunov inequality terms
A	System state coefficients matrix
A_d	Missile dynamics state coefficients matrix
α	Arbitrary positive constant
b	Input coefficients vector
b_d	Missile dynamics input coefficients vector
c	Output coefficients vector
c_d	Missile dynamics output coefficients vector
c_m	Constant coefficient ($cos\, \delta_n\, /v_c$)
c_v	Constant ($cos\, \delta_n\, /v_m$)
c_T	Constant coefficient ($cos\, \theta_n\, /v_c$)
γ	Path angle
δ	Missile lead angle
$\delta(\cdot)$	Dirac delta function
H	Solution of Lyapunov equation
I	Unit matrix
λ	Line of sight angle
λ_j	$j = 1..n$ eigenvalues of matrix H
n	General order parameter
N	Navigation gain
N'	Adjusted navigation gain
φ	Integration variable
r	Missile-target position vector
ρ	Missile-target range
$\mathbb{R}$	The real numbers domain
$\Re$	Real part indicator
s	Laplace variable
t	Time variable
T_e	Linear transformation matrix
t_f	Flight time
t_{go}	Time-to-go
τ_1	Arbitrary positive variable
τ_2	Arbitrary positive variable
τ_m	Missile dynamics time-constant
θ	Target aspect angle
u	Input signal
v	Velocity vector
v	Velocity magnitude
$v(\cdot)$	Lyapunov function
v_c	Missile-target closing velocity
v_j	$j = 1,2,3$ temporal auxiliary functions
x	System state vector
x_d	Missile dynamics state vector
x_e	Extended system state vector
y	Output signal
ω	Line of sight angle rate
ω_n	Natural frequency in 2^{nd} order system

z	Transformed system state vector
z_e	Extended equivalent state vector
ζ	Damping ratio in 2nd order system
Subscript 0	At $t = 0$
Subscript T	Of the target
Subscript m	Of the missile
Subscript n	Of the nominal trajectory
Subscripts $x/y/z$	At coordinate $x/y/$z
Overdot $\dot{}$	Derivative with respect to time
HOT	Higher Order Terms
LOS	Line Of Sight
LTI	Linear Time Invariant
LTV	Linear Time Varying
NCC	Near Collision Course
PN	Proportional Navigation
PPN	Pure Proportional Navigation
STT	Skid To Turn
TPN	True Proportional Navigation
ZEM	Zero Effort Miss

Abstract

In this work, we study the stability of a proportional navigation guidance system by using a set of analytical and numerical tools in order to provide a complete investigation of the guidance loop, from acceleration command to miss distance.

The guidance loop includes kinematics equations, guidance law and missile dynamics. The mathematical model that represents the system is nonlinear. This precise nonlinear model is the basis for any later analysis.

Linearization of the proportional navigation guidance system is done around a reference nominal process which is an ideal trajectory. With the linearized model a more in-depth research can be achieved, via application of theoretical analyses. The derived linearized model is time dependent such that the coefficients of the state matrices are factors of the remaining flight-time t_{go} (time-to-go).

Stability analysis of terminal-systems is an extension to the Lyapunov theory, applied in the guidance system to cope with time-varying components in the state-space model. The dependence of the model on t_{go} means that a singular point is present in the differential equation at the vicinity of the impact point ($t_{go} \rightarrow 0$). Then the stability of the terminal-systems, i.e. the subsystem that is governed by the differential equations of the line of sight angle rate $\omega(t)$ and the missile dynamics, is investigated in terms of unbounded increase of the flight time ($t_f \rightarrow \infty$). The results are sufficient conditions for stability and asymptotic stability.

To the extent that conditions regarding stability of the closed-loop subsystem exist, transition to the next step is admitted. The following investigation examines the stability of the subsystem that includes the missile-target range $\rho(t)$, whose final value forms the miss distance.

In regards to the purpose of the guidance system, the miss distance analysis is of paramount importance. Since two integrators are present at the output of the closed-loop subsystem, for which conditions for stability were provided earlier, then the stability of the subsystem that includes $\rho(t)$ requires study of its own. In contrary to the analysis of the closed-loop system, where stability is examined with respect to a prior state, the analysis of the miss distance is with respect to stability in the sense of input-output behavior, which is the response to an initial state. Numerical analysis examines the operation of the guidance system in this regard. Once again, the results provide sufficient conditions for stability and asymptotic stability.

An extra product was yielded through the Lyapunov analysis which can be used as a general tool for the design of the guidance system. The solution of Lyapunov function yields parametric inequality, which, by comparing it to the numeric objectives of the system, may indicate appropriate values for realization.

The paper is organized as follows: Chapter 1 covers the physical and mathematical background material required for study of the guidance loop. Chapter 2 is devoted to the system linearization. Chapters 3 and 4 present the closed-loop analysis (Lyapunov extension) and miss distance analysis of the guidance system respectively. In Appendix 1 is the program code of proportional navigation simulation.

1. Introduction to Proportional Navigation

This chapter provides a background for some of the main topics in the missiles guidance field. The guidance law of interest in this research paper is proportional navigation. Therefore, by the nature of things, the proportional navigation law also stands in the focus of this chapter. The kinematics of missile-target pursuit and elementary notions of this discipline are presented. Then we turn to study the proportional navigation law, the guidance circle and representations of missile dynamics.

1.1. Background

Three stages form the missile flight – launch, midcourse and terminal. In the first stage the missile accelerates until burnout and positions itself in a stable flight toward the target. In the second stage, the missile flies most of the distance to the target. It is common in this stage to attempt to increase hit performances by trajectory shaping. The final stage is known as the terminal stage.

In the last stage, accurate navigation should bring the missile close enough to the target to guarantee lethality. The navigation here is based on the relative motion of the missile and the target. The guidance system receives measures from a seeker and generates commands to be performed by the control system. The control system activates control surfaces motored by servos, to achieve the desired acceleration.

The resultant acceleration is in the lateral plane of the missile body (see Figure 1.1). Hence, the guidance system deals only with displacement and deviations in the two lateral axes of the missile.

The missile configuration in which a cylindrical body carries two sets of control surfaces, whether in symmetric form (90° separation between wings) or not (one set slightly rotated), is called skid-to-turn (*STT*).

Maneuvering in the perpendicular lateral axes, in skid-to-turn configuration is decoupled by means of roll stabilization and is managed in each channel independently.

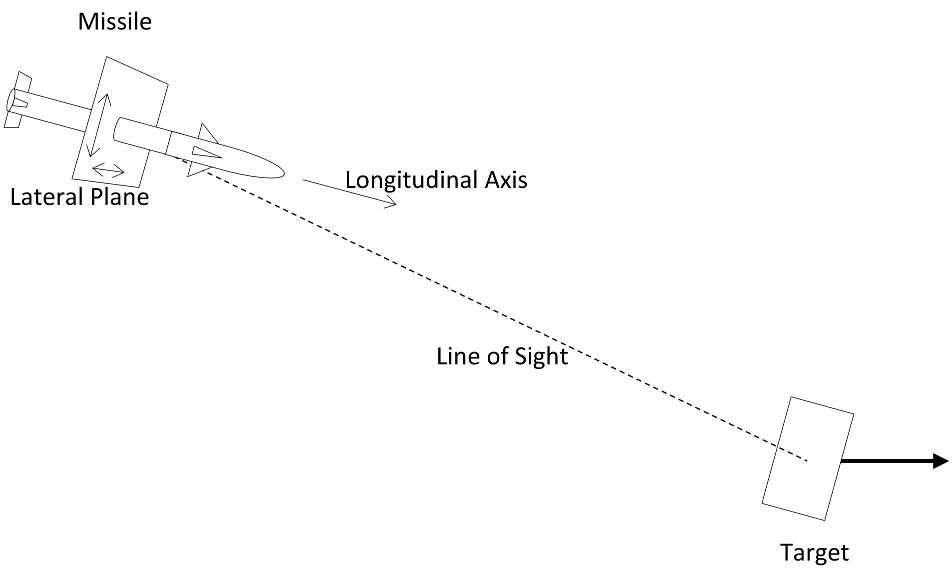

Figure 1.1: Guidance of a missile around the lateral axes in the Terminal stage.

Depending on the engaged method, the guidance law may be given in terms of required acceleration or velocity direction, finally the demand is translated into lateral force and wings deflection.

The Pure Pursuit method provides a guidance law that generates demands on velocity direction. In advance, a missile employing Pure Pursuit rotates its velocity direction to coincide along with the line of sight to the target.

For an optimal law, such as Proportional Navigation, the command is given with respect to the line of sight rate, as elaborated in the next sections.

1.2. Geometry and Kinematics

In this section properties and formulas related to the two-dimensional kinematics of missile guidance are presented [1] [2]. First, let the following assumptions:

a.1. The motion is two-dimensional. A common missile configuration is of two perpendicular channels, where the guidance of each channel is managed independently.

a.2. The Missile flight is in the post-boost phase. Energy is no longer consumed and momentum conservation is satisfied. Thus in the longitudinal axis the missile speed is remaining constant. The target speed is also assumed to be constant.

a.3. In the lateral plane acceleration (maneuver) is developed from guidance commands. Accelerations due to drag and gravity are ignored both in the longitudinal and lateral directions.

a.4. Missile dynamics is ideal, specifically:

$$a_m = a_c \tag{1.1}$$

Where a_c is the lateral acceleration command as delivered by the control system and a_m is the actual yielded lateral acceleration of the missile.

a.5. By referring always to the center of mass, the motion of the bodies is described via kinematics of points.

(Except assumption a.4, all the assumptions are valid also for the rest of the paper).

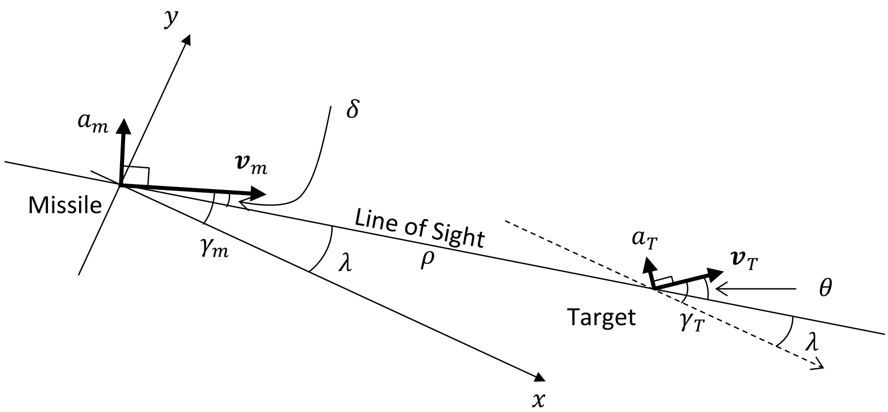

Figure 1.2: Kinematics of planar motion in a moment after first line of sight evaluation.

In the pursuit problem, the target and the missile position vectors are denoted by r_T and r_m respectively. $\mathrm{v_T}$ and $\mathrm{v_m}$ are the target and the missile velocity vectors with constant magnitudes v_T and v_m, and with directions determined by the path angles γ_T and γ_m, respectively.

The line that connects r_T and r_m is the Line of Sight (*LOS*) and its vector representation is r:

$$r = r_T - r_m \tag{1.2}$$

Where r is the line of sight vector, r_T and r_m are the target position vector and the missile position vector respectively.

The length of r is the range ρ and its angle is the LOS angle λ. λ is measured with respect to an inertial reference axis, usually fixed to the direction of the first line of sight measure (see Figure 1.2).

Elementary mechanics provides us with two velocities of planar motion – radial velocity:

$$\dot{\rho} = v_T \cdot cos\,\theta - v_m \cdot cos\,\delta \tag{1.3}$$

That is the relative velocity along the line of sight – and the normal velocity:

$$\rho \cdot \dot{\lambda} = v_T \cdot sin\,\theta - v_m \cdot sin\,\delta \tag{1.4}$$

That is the relative velocity perpendicular to the line of sight.

Where $\dot{\rho}$ and $\dot{\lambda}$ are the first time-derivative of the LOS range and the LOS angle respectively. v_T and v_m are the constant magnitudes of the target and the missile velocities. θ is the target aspect angle that formed by the target velocity and the LOS and δ is the missile lead angle that formed by the missile velocity and the LOS.

For the Cartesian frame of coordinates taken as fixed and inertial, the components of the line of sight vector r are the projections of ρ through λ:

$$\begin{aligned} r_x &= \rho \cdot cos\,\lambda \\ r_y &= \rho \cdot sin\,\lambda \end{aligned} \tag{1.5}$$

Where r_x is the component of the line of sight vector r on the x axis, r_y is the component of the line of sight vector r on the y axis, ρ is the magnitude of r and λ is its angle with x axis.

The acceleration normal to the path in non-uniform motion, yields the terms for the target and the missile path angles:

$$\begin{aligned} \dot{\gamma}_T &= a_T / v_T \\ \dot{\gamma}_m &= a_m / v_m \end{aligned} \tag{1.6}$$

Where γ_T is the target path angle, γ_m is the missile flight path angle, a_T target acceleration perpendicular to the target velocity, a_m missile acceleration perpendicular to the missile velocity, v_T and v_m are the constant magnitudes of the target and the missile velocities.

The angles θ and δ that appear in (1.3) and (1.4) are known as the aspect angle and the lead angle. θ formed by the target velocity and the line of sight and δ by the missile velocity and the line of sight (Figure 1.2):

$$\begin{aligned} \theta &= \gamma_T - \lambda \\ \delta &= \gamma_m - \lambda \end{aligned} \tag{1.7}$$

Where θ is the target aspect angle and δ missile lead angle, γ_T and γ_m are target and missile path angles respectively, λ *LOS* angle.

For the missile to be on a collision course with the target (see section 2.1 in chapter 2), the law of sines determines that the lead angle has to satisfy:

$$\delta = sin^{-1}\left(\frac{v_T}{v_m} \cdot sin\,\theta\right) \tag{1.8}$$

Where δ missile lead angle, θ target aspect angle, v_T and v_m are the target and the missile constant velocities.

In practice the lead angle is different due to instruments limitations and biases. In this case the lead angle is separated into two components, one is said to be the correct lead angle and the second the heading error:

$$\delta = sin^{-1}\left(\frac{v_T}{v_m} \cdot sin\,\theta\right) + \delta_{err} \tag{1.9}$$

Where δ missile lead angle, θ target aspect angle, v_T and v_m are the target and the missile constant velocities, δ_{err} is the heading error.

The guidance effectiveness of a pursuit is usually evaluated with respect to the initial heading error δ_{err}.

Let the *LOS* angle rate $\dot{\lambda} = \omega$ and substitute (1.7) for θ and δ in (1.4):

$$\rho \cdot \omega = v_T \cdot sin(\gamma_T - \lambda) - v_m \cdot sin(\gamma_m - \lambda) \tag{1.10}$$

Where ρ is the *LOS* range and ω the *LOS* angle rate, λ *LOS* angle, v_T and v_m are the target and the missile constant velocities, γ_T is the target path angle, γ_m is the missile flight path angle.

Differentiation of both sides with respect to time yields:

$$\begin{aligned} \rho \cdot \dot{\omega} + \dot{\rho} \cdot \omega = {} & v_T \cdot \dot{\gamma}_T \cdot cos(\gamma_T - \lambda) - v_m \cdot \dot{\gamma}_m \cdot cos(\gamma_m - \lambda) \\ & + \underbrace{\dot{\lambda} \cdot v_m \cdot cos(\gamma_m - \lambda) - \dot{\lambda} \cdot v_T \cdot cos(\gamma_T - \lambda)}_{-\omega \cdot \dot{\rho}} \end{aligned} \tag{1.11}$$

Where ρ is the *LOS* range and ω the *LOS* angle rate, λ *LOS* angle, v_T and v_m are the target and the missile constant velocities, γ_T is the target path angle, γ_m is the missile flight path angle.

The two last terms in (1.11) sum to $-\omega \cdot \dot{\rho}$, the expressions $v_T \cdot \dot{\gamma}_T$ and $v_m \cdot \dot{\gamma}_m$ are equal a_T and a_m as in (1.6), hence the first time-derivative of the *LOS* angle rate gets the last form:

$$\dot{\omega} = \frac{1}{\rho}[-2 \cdot \omega \cdot (v_T \cdot cos\,\theta - v_m \cdot cos\,\delta) + a_T \cdot cos\,\theta - a_m \cdot cos\,\delta] \qquad (1.12)$$

Where ω is the missile-target angular velocity (*LOS* rate), ρ is the *LOS* range, a_T and a_m are the target and the missile accelerations, θ is the target aspect angle, δ missile lead angle.

Having developed the self-contained equations of motion for the missile-target pursuit, the total system is now given by the following set of differential equations:

$$\begin{aligned}
&\dot{\rho} = v_T \cdot cos\,\theta - v_m \cdot cos\,\delta \\
&\dot{\lambda} = \omega \\
&\dot{\omega} = -2 \cdot \omega \cdot (v_T \cdot cos\,\theta - v_m \cdot cos\,\delta)/\rho + a_T \cdot cos\,\theta\,/\rho \\
&\qquad\qquad - a_m \cdot cos\,\delta\,/\rho \\
&\dot{\gamma}_T = a_T/v_T \\
&\dot{\gamma}_m = a_m/v_m
\end{aligned} \qquad (1.13)$$

Where ρ is the line of sight range, λ is the line of sight angle, ω is the line of sight angle rate, v_T target constant velocity magnitude, v_m missile constant velocity magnitude, γ_T target path angle, γ_m missile flight path angle, θ is the target aspect angle and δ missile lead angle, a_T and a_m are the accelerations of the target and the missile respectively.

Notes on (1.13):

n.1. Equations (1.13) form a complete mathematical model of the relative motion of missile and target in two dimensions subject to restrains a.1 – a.5.

n.2. Evaluation of (1.13) have to be preceded by the calculation of θ and δ by the algebraic equations (1.7).

n.3. Expressions for initial conditions embedded in previous equations. ρ_0 and λ_0 are given by the initial magnitude of the *LOS* vector r and its angle with x axis. ω_0 is calculated by (1.4). Initial condition for $\gamma_T = \theta_0 + \lambda_0$ entails knowledge of the target velocity vector. For the missile path angle, $\gamma_{m_0} = \delta_0 + \lambda_0$, (1.8) has to be calculated.

1.3. Proportional Navigation

So far the equations of motion and the geometry with respect to the missile guidance were described, but not much was said about the law by which the missile may hit the target. This is the place where the Proportional Navigation (PN) is introduced and provides a guidance law to lead the missile towards collision with the target [3].

Proportional navigation seeks to preserve a constant line of sight angle and thus to place the missile in a collision course with the target. By providing the control system a reference signal proportional to the line of sight rate, the guidance system rotates the missile velocity vector with the line of sight. Thus an acceleration command form of the PN law is [2]:

$$a_c = N \cdot v_m \cdot \omega \tag{1.14}$$

Where a_c is a lateral acceleration command needed to perform by the control system, N constant navigation gain, v_m missile velocity and ω the line of sight angle rate.

With respect to the pursuit model (1.13), if the missile dynamics is assumed ideal (1.1) then a_c is replacing a_m directly. If nonideal dynamics is considered, then a_c is standing for the missile acceleration $a_{m,}$, through the model that represents the missile dynamics. In any case, since the acceleration command (1.14) is algebraic with respect to ω, its calculation has to come before the calculation of the differential equations (1.13) (see also note n.2).

The type of law presented in (1.14), and will be used further on in this work, is called Pure Proportional Navigation (*PPN*) and in it commands are applied perpendicular to the missile velocity (see Figure 1.2). Another common approach is the True Proportional Navigation (*TPN*) in which commands are given with respect to the line of sight.

Experiments show that values for the navigation gain N (1.14) are preferred when taken between 3 and 5, and in any way, no lower than 2. The higher the gain is, the flight path is more gentle and slightly closer to the region bounded around the initial conditions. However a higher gain means also large maneuvering capabilities [3].

Proportional navigation with $N = 3$ demonstrated to be the optimal solution for the pursuit problem regarding non-maneuvering targets, when the cost function is minimum miss distance [4]. The development of guidance laws with respect to optimal control, is currently the subject of a large study with the purpose of providing solutions for modern problems.

Among other problems, there are the interceptions inside and outside the atmosphere. A work that studies exo-atmospheric pursuit achieves solutions [5] by associating a terminal cost function with the constrained kinematics. The cost function is defined in the sense of miss distance for a prescribed final time t_f (see section 2.1). Solution to the optimal problem yields guidance strategy in terms of zero effort miss (*ZEM*), namely the miss that would be obtained by stopping to produce commands from the current to the final time.

A study of endo-atmospheric interception for a missile having nonlinear dynamics and Gaussian measurement noise [6], proposes an integrated estimation-guidance approach on the basis of numerically solving the Hamilton-Jacobi equation associated with the stochastic optimization problem.

Another paper [7] studies the advantage of optimal guidance laws with respect to traditional ones given mismatches between the actual dynamics of the interceptor and the model in design. The results are given in terms of stability, in the time domain (Lyapunov), and the frequency domain (circle criterion).

The following sections deal with issues concerning missile guidance as a system.

1.4. The Guidance Loop

The complete missile is composed of several subsystems. Each subsystem is characterized by unique dynamics such as damping-ratio or time-constant. Analyses of missile guidance have to consider these properties in order to correctly model the system. Only then can the right approach to analyze the system be determined. Accordingly, the guidance loop is presented here in its full scale, namely a nonlinear model with nonideal dynamics.

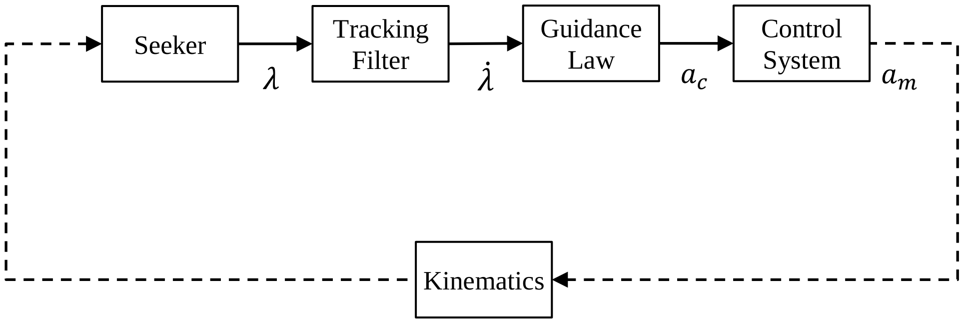

Figure 1.3: Proportional navigation guidance loop.

Target detection by the homing head – seeker – initiates the guidance loop. According to (1.14) proportional navigation requires information about the line of sight angular rate to produce guidance commands. To generate this datum, readings of the seeker are delivered to the tracking filter to estimate the line of sight angle and its derivative. The guidance law develops lateral acceleration commands. The flight control system activates control surfaces to force the missile to track the commands. Each cycle results in missile motion. The achieved motion alters the relative position between the missile and the target. This loop-action continues up until final conditions [1] are met.

The guidance system in Figure 1.3 can also be presented in a state-manner:

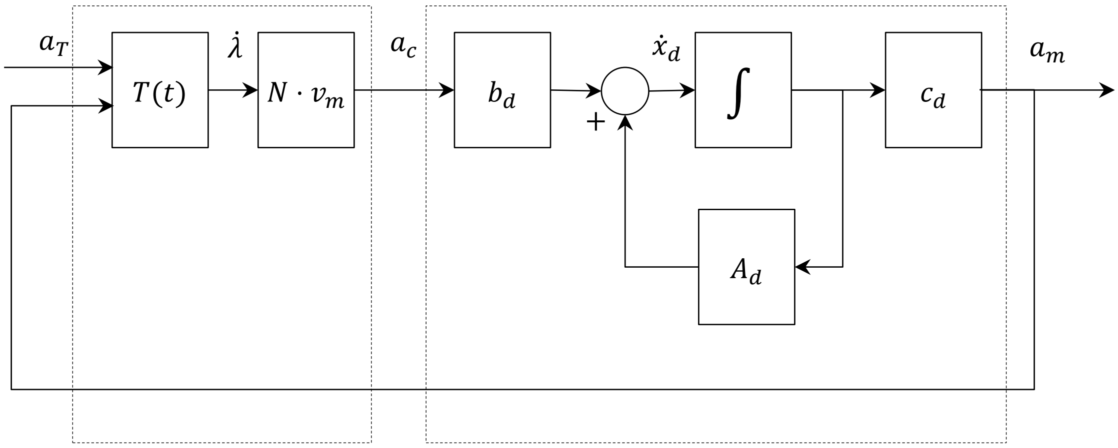

Figure 1.4: Concrete description of a guidance loop composed of guidance law block (left dashed box) and missile dynamics block (right).

It is convenient to distinguish between the two blocks of the guidance loop: one block includes the guidance components, and the other one represents the missile dynamics.

1.4.1. Guidance Law Block

The guidance law block includes the relation of the accelerations to the line of sight, as it is reflected in the equations of motion, and therefore it is inherently nonlinear time-dependent (expressed in Figure 1.4 with the operator $T(t)$).

Recall the kinematic formulation that connects a missile motion with the line of sight angle rate (1.11):

$$\rho \cdot \dot{\omega} + 2 \cdot \dot{\rho} \cdot \omega = a_T \cdot cos\,\theta - a_m \cdot cos\,\delta \qquad (1.11)$$

Where ρ is the line of sight range, ω is the line of sight angle rate, θ is target aspect angle and δ missile lead angle, a_T and a_m are the lateral accelerations of the target and the missile respectively.

Since the variables ρ, ω, θ and δ change quickly with time, they form a relation between the input and the output of the guidance block, that depends on time. Then $T(t)$ in Figure 1.4, is an operator that connects the missile acceleration and the LOS rate. The expression for $T(t)$ may be found in the rearrangement of (1.11).

Important to notice that since a missile objective is to close the range as fast as possible, the variables that take place at the guidance law block change quickly in a short and finite time, and indeed the guidance system as a whole is characterized by a dependence on regression of finite time, as will be discussed later.

As mentioned, the guidance block stands on the kinematical relation as reflected by the equations of motion. Therefore as a model this block is dominated by the five variables of equations (1.13). For convenience, let us write this set of equations here again:

$$\begin{aligned}
&\dot{\rho} = v_T \cdot \cos\theta - v_m \cdot \cos\delta \\
&\dot{\lambda} = \omega \\
&\dot{\omega} = -2 \cdot \omega \cdot (v_T \cdot \cos\theta - v_m \cdot \cos\delta)/\rho + a_T \cdot \cos\theta\,/\rho \\
&\qquad\qquad - a_m \cdot \cos\delta\,/\rho \\
&\dot{\gamma}_T = a_T/v_T \\
&\dot{\gamma}_m = a_m/v_m
\end{aligned} \tag{1.13}$$

Where ρ the line of sight range, λ is the line of sight angle, ω is the line of sight angle rate, v_T target constant velocity magnitude, v_m missile constant velocity magnitude, γ_T target path angle, γ_m missile flight path angle, θ is target aspect angle and δ missile lead angle, a_T and a_m are the accelerations of the target and the missile respectively.

Then the guidance block is composed of five variables $[\rho \quad \lambda \quad \omega \quad \gamma_T \quad \gamma_m]$.

1.4.2. Missile Dynamics Block

Missile dynamics includes autopilot, seeker and tracking filters, all these subsystems' coefficients are mostly constant, and therefore may be represented by a linear time-invariant (LTI) model.

Although each of the subsystems, namely autopilot, seeker and tracking filters, justifies a transfer function of some order for its own, it is common to study the effects of the dynamics by considering an overall, one linear transfer function. It transfers the desired acceleration a_c, as produced by the guidance unit, to actual missile acceleration a_m, as performed after the course that was done from the line of sight measure and until the fins deflection and the consequent lift force, that changes the missile lateral acceleration.

The general form of transfer function for dynamics of n^{th} order is given by [8]:

$$a_m = \frac{b_{n-1} \cdot s^{n-1} + \cdots + b_1 \cdot s + 1}{a_n \cdot s^n + a_{n-1} \cdot s^{n-1} + \cdots + a_1 \cdot s + 1} \cdot a_c \tag{1.15}$$

Where a_m is the missile actual achieved lateral acceleration, a_c lateral acceleration command as given by the navigation law, s Laplace transform variable, a_i coefficients of the characteristic polynomial of the transfer function and b_i coefficients of the numerator, n is the number of the highest exponent of the characteristic polynomial.

The missile dynamics represented in (1.15) by a transfer function, may also be represented by the following state space form (Figure 1.4):

$$\begin{gathered} \dot{x}_d = A_d \cdot x_d + b_d \cdot a_c \\ a_m = c_d \cdot x_d \end{gathered} \tag{1.16}$$

Where $A_d \in R^{n\times n},\ b_d \in R^{n\times 1},\ c_d \in R^{1\times n}$ are state matrix and vectors composed of the coefficients a_i and b_i of the transfer function (1.15), $x_d \in R^{n\times 1}$ state vector of the dynamic variables, a_c and a_m are input and output lateral acceleration respectively. The equation is subject to the initial conditions $x_d = x_{d_0}$ at $t = t_0$.

The variables that dominate this block of the system are dependent on the model that represents the dynamics of the missile. It may begin with a first order model where only one variable, that is the missile lateral acceleration, is considered, and go to a high arbitrary order of the missile dynamic states [9]. The number of variables that represent the dynamics block is denoted by n.

In order to maintain the gain of the guidance law (see section 1.3) the different components of the missile dynamics have to be matched in such a way that the overall gain from input to output will be:

$$\frac{a_m}{a_c} = c_d \cdot (s \cdot I - A_d)^{-1} \cdot b_d|_{s=0} = 1 \tag{1.17}$$

Where a_c missile lateral acceleration input command and a_m missile lateral output acceleration. $A_d \in R^{n\times n},\ b_d \in R^{n\times 1},\ c_d \in R^{1\times n}$ are state matrix and vectors of the missile dynamics, s Laplace variable and I is unit matrix of order n.

When an ideal dynamics assumption is made, the connection between the input acceleration command and the output actual acceleration is direct, that is to say A_d, b_d and c_d reset and a second route connects a_m to a_c through a variable d_d, which equals 1. As stated in (1.1):

$$a_m = a_c \tag{1.1}$$

Where a_c missile lateral acceleration command and a_m missile actual achieved lateral acceleration.

1.4.3. State Space Model

To describe the model of the guidance system in more detail, let the state variables be:

$$\begin{aligned} x_1 &= \rho \\ x_2 &= \lambda \\ x_3 &= \omega \\ x_4 &= \gamma_T \\ x_5 &= \gamma_m \end{aligned} \tag{1.18}$$

Where $[x_1, \dots, x_5] = [\rho, \lambda, \omega, \gamma_T, \gamma_m]$ are state variables of the system, ρ line of sight range, λ line of sight angle, ω *LOS* angle rate, γ_T and γ_m path angles of target and missile respectively.

The variables (1.18) with the missile dynamics vector form the state vector x of the guidance system:

$$x = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_d]^T \tag{1.19}$$

Where $x \in R^{(5+n)\times 1}$ is the guidance system state vector, $[x_1, \dots, x_5] = [\rho, \lambda, \omega, \gamma_T, \gamma_m]$ are the state variables of motion as detailed in (1.18), $x_d \in R^{n\times 1}$ is the state vector of the missile dynamics (1.16), superscript T refers to the transpose operator.

The number of states $5 + n$ in the vector (1.19) originates in the five variables of motion of the guidance block, plus n variables of the missile dynamics (see previous sections).

Putting together (1.13) and (1.16), the complete model of the guidance system with state vector x (1.19) receives its final form:

$$\begin{aligned} \dot{x}_1 &= v_T \cdot \cos\theta - v_m \cdot \cos\delta \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -2 \cdot x_3 \cdot (v_T \cdot \cos\theta - v_m \cdot \cos\delta)/x_1 + a_T \cdot \cos\theta / x_1 \\ &\qquad - c_d \cdot x_d \cdot \cos\delta / x_1 \\ \dot{x}_4 &= a_T / v_T \\ \dot{x}_5 &= c_d \cdot x_d / v_m \end{aligned} \tag{1.20}$$

$$\dot{x}_d = A_d \cdot x_d + b_d \cdot a_c$$

Where $x \in R^{(n+5)\times 1}$ is the guidance system state vector, $[x_1, \dots, x_5] = [\rho, \lambda, \omega, \gamma_T, \gamma_m]$ are the state variables of motion, $x_d \in R^{n\times 1}$ is the state vector of the missile dynamics with state matrix $A_d \in R^{n\times n}$ and vectors $b_d \in R^{n\times 1}$, $c_d \in R^{1\times n}$, v_T and v_m target and missile constant velocities, θ and δ target aspect angle and missile lead angle, a_T is the target acceleration and a_c missile acceleration command.

Recall also that the calculation of the right hand side of (1.20) must be preceded by the evaluation of θ, δ and a_c by (1.7) and (1.14):

$$\begin{aligned} \theta &= \gamma_T - \lambda \\ \delta &= \gamma_m - \lambda \\ a_c &= N \cdot v_m \cdot \omega \end{aligned} \tag{1.21}$$

Where θ is the target aspect angle and δ missile lead angle, γ_T and γ_m are target and missile path angles respectively, λ LOS angle, a_c is an acceleration command needed to perform by the control system, N constant navigation gain, v_m missile velocity and ω the line of sight angle rate.

The variables θ, δ and a_c of the algebraic equations (1.21) form the vector x_a:

$$x_a = [\theta \quad \delta \quad a_c]^T \tag{1.22}$$

Where $x_a \in R^{3\times 1}$ is the vector of the dependent variables of algebraic equations (1.21), θ is the target aspect angle and δ missile lead angle, a_c missile acceleration command, superscript T indicates the transpose operator.

Initial conditions to the motion variables in (1.20) are collected from the equivalent formulations, as described at note n.3 of section 1.2. Let's add the initial conditions of the dynamics, x_{d_0}, and the result is the set of initial conditions required for the calculation of the state equations (1.20):

$$\begin{aligned} &\rho_0 = \sqrt{r_{x_0}^2 + r_{y_0}^2} \\ &\lambda_0 = sin^{-1}(r_{y_0}/\rho) \\ &\omega_0 = (v_T \cdot sin\,\theta_0 - v_m \cdot sin\,\delta_0)/\rho_0 \\ &\gamma_{T_0} = \theta_0 + \lambda_0 \\ &\gamma_{m_0} = sin^{-1}(sin(\theta_0) \cdot v_T/v_m) + \delta_{err} + \lambda_0 \\ &x_{d_0} = given \end{aligned} \tag{1.23}$$

Where subscript 0 stands for initial time t_0, ρ is the line of sight range, r_x and r_y are the components of the relative position r in $\hat{x}$ and $\hat{y}$ directions respectively, λ is the line of sight angle, ω line of sight angle rate, v_T and v_m target and missile velocities, γ_T and γ_m target and missile path angles, θ target aspect angle and δ missile lead angle, δ_{err} initial heading error – the deviation of the missile lead angle from the correct direction (see 1.8), x_d state vector of missile dynamics of n^{th} order.

Finally let f be a functions-vector that represents the right hand side of the ordinary nonlinear differential equations (1.20):

$$\dot{x} = f(x, a_c, a_T) \tag{1.24}$$

Where $x, \dot{x} \in R^{(5+n)\times 1}$ are the guidance system state vector and its derivative, $f \in R^{(5+n)\times 1}$ is a vector of functions detailed in (1.20), a_T is the target acceleration and a_c missile acceleration command.

In the same manner $f_a(x_a)$ represents the right hand side of the algebraic equations (1.22).

Equations (1.20) are highly nonlinear. Due to the analyzation difficulty of (1.20), a design objective is to linearize the equations. In the next section, (1.20) will be expanded into a Taylor series about a nominal trajectory at the final stage of a missile's course, where proportional navigation is engaged.

1.5. Nonlinear System – Examples

For verification and demonstration of the results here and in later examples, we are going to use three sets of missile dynamics and parameters. In all of them the nonlinear *2D* kinematic model is engaged. The three types of dynamics are the ideal, first order and second order dynamics. The value of the following parameters is fixed unless mentioned explicitly:

$v_m = 400\ m/s$, $v_T = 200\ m/s$, $a_T = 0$, $N = 3$.

Ideal Dynamics

Algebraic equations:

$$\begin{aligned} \theta &= \gamma_T - \lambda \\ \delta &= \gamma_m - \lambda \\ a_c &= 400 \cdot N \cdot \omega \end{aligned} \tag{1.25}$$

Differential equations:

$$
\begin{aligned}
\dot{x}_1 &= 200 \cdot cos\,\theta - 400 \cdot cos\,\delta \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= [-2 \cdot x_3 \cdot (200 \cdot cos\,\theta - 400 \cdot cos\,\delta) - a_c \cdot cos\,\delta]/x_1 \\
\dot{x}_4 &= 0 \\
\dot{x}_5 &= a_c/400
\end{aligned} \tag{1.26}
$$

Initial conditions for the differential equations:

$$
\begin{aligned}
x_{1_0} &= x_T \\
x_{2_0} &= 0 \\
x_{3_0} &= (200 \cdot sin\,\theta_0 - 400 \cdot sin\,\delta_0)/x_T \\
x_{4_0} &= \theta_0 \\
x_{5_0} &= \delta_0
\end{aligned} \tag{1.27}
$$

Initial condition values are provided on the basis of the following evaluations:

$$
\begin{aligned}
&\rho_0 = \sqrt{r_{x_0}^2 + r_{y_0}^2} = \sqrt{\left(x_{T0} - x_{m0}\right)^2 + \left(y_{T0} - y_{m0}\right)^2} = x_{T0} \\
&\lambda_0 = sin^{-1}(r_{y_0}/\rho) = 0 \\
&\omega_0 = (v_T \cdot sin\,\theta_0 - v_m \cdot sin\,\delta_0)/\rho_0 = (200 \cdot sin\,\theta_0 - 400 \cdot sin\,\delta_0)/x_{T0} \\
&\gamma_{T_0} = \theta_0 + \lambda_0 = \theta_0 \\
&\gamma_{m_0} = sin^{-1}(\cdot\, v_T/v_m\, sin\,\theta_0) + \delta_{err} + \lambda_0 = sin^{-1}\left(\frac{1}{2} \cdot sin\,\theta_0\right) + \delta_{err}
\end{aligned} \tag{1.28}
$$

$\lambda_0 = sin^{-1}(r_{y_0}/\rho) = 0$ is a result of the selection of a frame with x axis aligned with the initial *LOS* (see Figure 1.2, Figure 1.5).

1st Order Dynamics

Based on the first order transfer function:

$$
a_m = \frac{1}{\tau_m \cdot s + 1} \cdot a_c \tag{1.29}
$$

Where a_m is the missile actual acceleration at the lateral plane, a_c *PN* command, s Laplace transform variable and τ_m time-constant of the missile dynamics.

The following equations are valid, algebraic equations:

$$
\begin{aligned}
\theta &= \gamma_T - \lambda \\
\delta &= \gamma_m - \lambda \\
a_c &= 400 \cdot N \cdot \omega
\end{aligned} \tag{1.30}
$$

Differential equations:

$$\begin{aligned}
\dot{x}_1 &= 200 \cdot \cos\theta - 400 \cdot \cos\delta \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= [-2 \cdot x_3 \cdot (200 \cdot \cos\theta - 400 \cdot \cos\delta) - x_6 \cdot \cos\delta]/x_1 \\
\dot{x}_4 &= 0 \\
\dot{x}_5 &= x_6/400 \\
\dot{x}_6 &= -x_6/\tau_m + a_c/\tau_m
\end{aligned} \tag{1.31}$$

Differential equations initial conditions (see calculations in 1.27):

$$\begin{aligned}
x_{1_0} &= x_T \\
x_{2_0} &= 0 \\
x_{3_0} &= (200 \cdot \sin\theta_0 - 400 \cdot \sin\delta_0)/x_T \\
x_{4_0} &= \theta_0 \\
x_{5_0} &= \delta_0 \\
x_{6_0} &= a_{m_0}
\end{aligned} \tag{1.32}$$

2nd Order Dynamics

Based on the second order transfer function:

$$a_m = \frac{\omega_n^2}{s^2 + 2 \cdot \zeta \cdot \omega_n \cdot s + \omega_n^2} \cdot a_c \tag{1.33}$$

Where a_m is the missile actual acceleration at the lateral plane, a_c PN command, s Laplace transform variable. ζ is the damping ratio and ω_n is the natural frequency of the missile dynamics.

Algebraic equations:

$$\begin{aligned}
\theta &= \gamma_T - \lambda \\
\delta &= \gamma_m - \lambda \\
a_c &= 400 \cdot N \cdot \omega
\end{aligned} \tag{1.34}$$

Differential equations:

$$\begin{aligned}
\dot{x}_1 &= 200 \cdot \cos\theta - 400 \cdot \cos\delta \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= [-2 \cdot x_3 \cdot (200 \cdot \cos\theta - 400 \cdot \cos\delta) - x_6 \cdot \cos\delta]/x_1 \\
\dot{x}_4 &= 0 \\
\dot{x}_5 &= x_6/400 \\
\dot{x}_6 &= x_7 \\
\dot{x}_7 &= -\omega_n^2 \cdot x_6 - 2 \cdot \zeta \cdot \omega_n \cdot x_7 + \omega_n^2 \cdot a_c
\end{aligned} \tag{1.35}$$

Differential equations initial conditions (see calculations in 1.27):

$$\begin{aligned}
x_{1_0} &= x_T \\
x_{2_0} &= 0
\end{aligned} \tag{1.36}$$

$x_{3_0} = (200 \cdot sin\,\theta_0 - 400 \cdot sin\,\delta_0)/x_T$
$x_{4_0} = \theta_0$
$x_{5_0} = \delta_0$
$x_{6_0} = a_{m_0}$
$x_{7_0} = x_{d_1}$

Example 1 – Initial Heading Error

The examples in this section examine the guidance system response for two types of challenges in initial conditions: heading error and target range. The ideal system and first order system with the governing equations (1.25)-(1.32) are used.

Let the missile, located at the origin, employ proportional navigation. The target is not maneuvering, that is $a_T = 0$. Additional details appear in Figure 1.5.

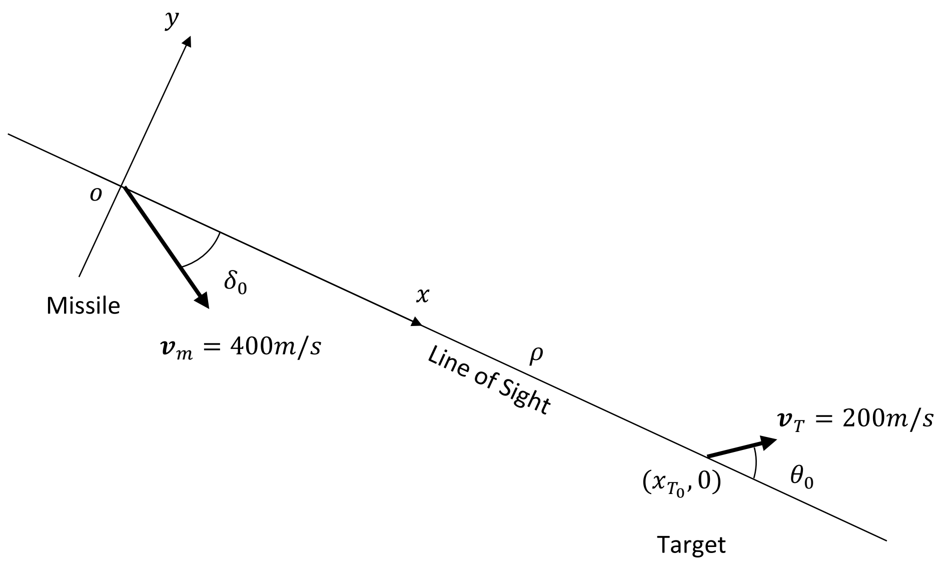

Figure 1.5: Initial conditions and parameters for the examples in 1.5.

Figure 1.6 is the acceleration command versus initial heading errors δ_{err_0} and $2 \cdot \delta_{err_0}$. The results imply that for the higher heading error (gray scales in the figures) the required acceleration to close the error in the velocity direction, is two times higher than the equivalent lower heading error.

The figure presents also results for nonideal dynamics, which is simulated by a first order transfer function, in this case the achieved acceleration (a_m) is not ideally as the acceleration command (a_c). For the two cases of initial heading error, it is possible to see the actual missile acceleration marked by a dashed line.

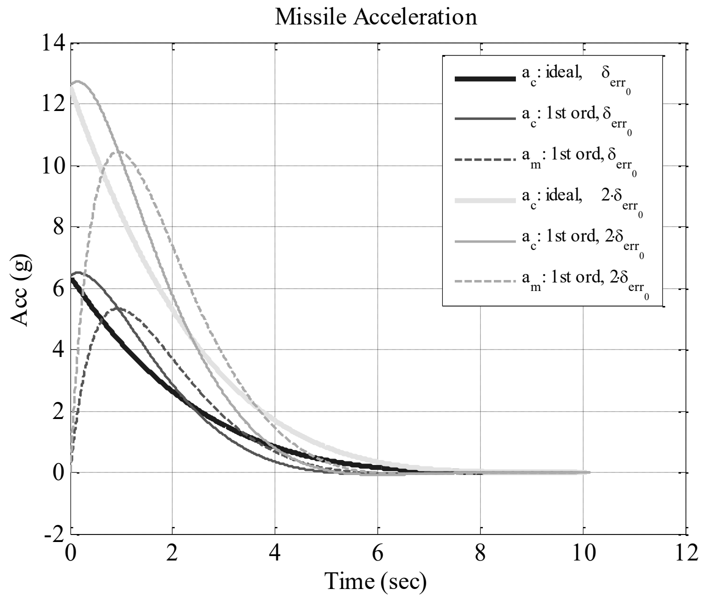

Figure 1.6: Acceleration histories of guidance system for two values of initial heading error δ_{err_0} , for ideal system and for first order dynamics.

Figure 1.7 is the trajectory extracted from the relative position vector r. In terms of time series, the line begins at the right lower corner where the distance between the missile and the target is the largest. Then the distance advances and decreases until the closing velocity v_c changes its sign. At that time, the last range between the missile and the target is inscribed and the miss-distance result declared.

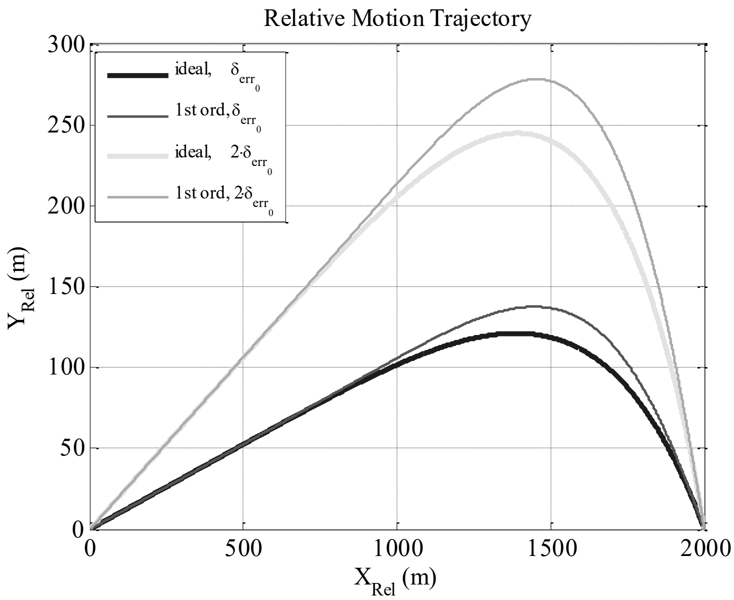

Figure 1.7: Relative motion trajectories for two values of initial heading error δ_{err_0} , for ideal system and for first order dynamics.

The effect of the initial heading error takes form in the convex curve of the trajectory plots of the bigger heading error.

Example 2 – Long Target Range

Figure 1.8 compares the acceleration commands of a system with an initial range ρ_0 and a system with a two times bigger initial range, $2 \cdot \rho_0$.

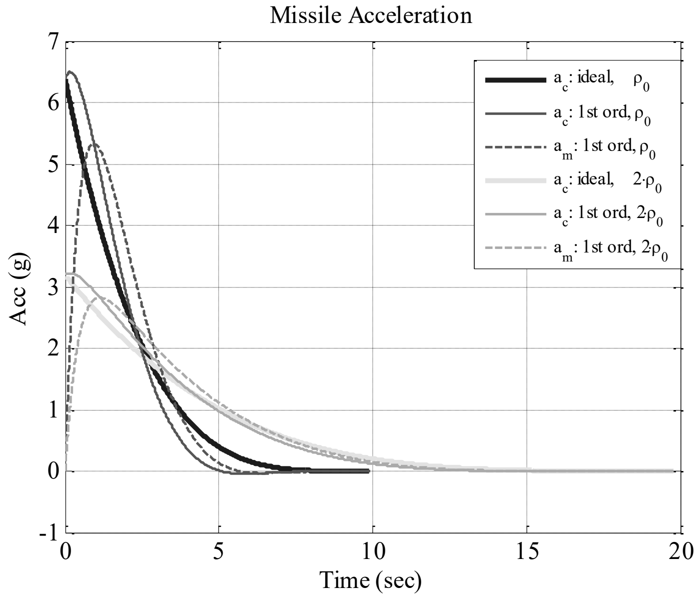

Figure 1.8: Acceleration histories of a guidance system for two values of initial range ρ_0, for ideal system and for first order dynamics.

The initial heading error δ_{err_0} for the two cases is the same. It is obvious that the flight length t_f will be about two times longer for the longer range, hence the time base for the cases is different. But it is possible to see that when the guidance system has a longer time to close the error in the velocity vector, it allows lower energy consumption in the sense of acceleration command. This is explained by the fact that the command is proportional to the *LOS* rate. Since the change in *LOS* is slower for a remote target, the command is lower as well.

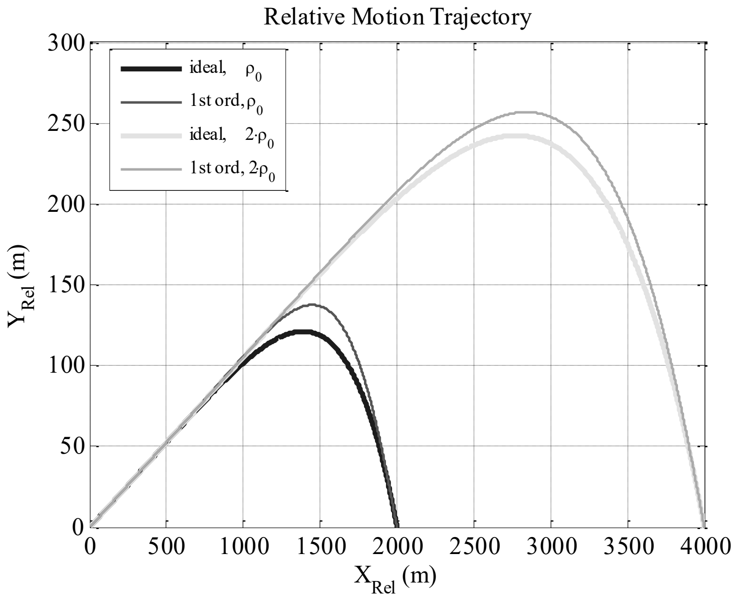

Figure 1.9: Relative motion trajectories for two values of initial range ρ_0, for ideal system and for first order dynamics.

The trajectories in Figure 1.9 are straightforward. The bigger range ends in high values of trajectories for both x and y axis. The high value of y is the result of a low command that enables a longer time to correct the velocity vector error.

1.6. Summary

Chapter 1 covered basic topics of the guidance as a preface for the study of the system under test. Laws of motion of the two-dimensional proportional navigation are introduced. Together with state space model of the missile dynamics, they were applied to conclude the nonlinear model of the guidance system. The equations of the model serve for system simulation as was presented with several examples in the last section.

2. System Linearization

The exact nonlinear model of the guidance system that was developed in the previous chapter is the basis for any proportional navigation study. First, as a mathematical model for numerical simulation, second, as simplification and approximation for an analytical study. A kind of approximation is the system linearization to be performed in the following chapter. Analyses in later chapters will use the linearized model.

2.1. Near Collision Course

State space representation of the exact nonlinear model of the guidance system developed in the previous chapter and concluded in differential equations (1.20), algebraic equations (1.21) and initial conditions (1.23). The complex form of the equations makes the closed-form solution impossible, and analysis in terms of stability or intercept-operations also are very difficult. In advance, in most of the researches the problem is simplified by assuming some assumptions or limiting the conditions over a specific domain. An additional crucial step is to linearize the system. In general, linearization is made about an equilibrium point, but as the solution of system (1.20) may involve time-varying elements, linearization here has to be performed with respect to a solution which is a nominal trajectory [10].

Consider collision course conditions:

c.1. Target velocity is constant in magnitude and direction. A realization of this condition is setting the target acceleration a_T to zero.

c.2. Missile velocity direction is ideally towards impact point with the target. In other words, lead angle δ (1.8) has no heading error component ($\delta_{err} = 0$).

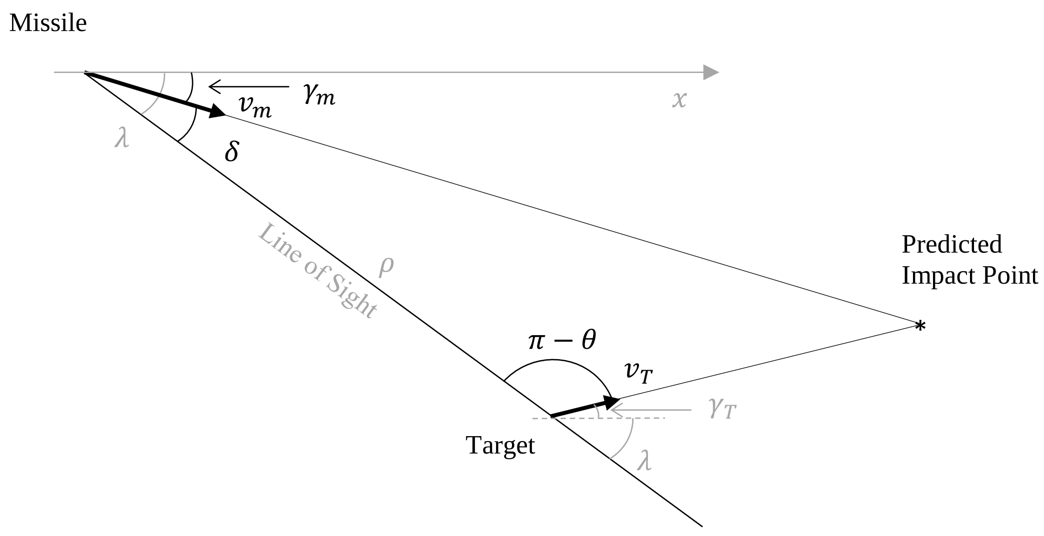

Figure 2.1: Kinematics of motion in a collision triangle: the missile lead angle δ oriented precisely to impact point.

Special attention is granted to conditions c.1 and c.2 because they imply a phase of flight in which no effort is required by the missile to meet the target. They are also important because the trajectories of the bodies are ultimately linear. The two trajectories form a geometric triangle whose third side is the instantaneous line-of-sight range (Figure 2.1).

A scenario that is not absolutely ideal as c.1 and c.2 but has properties that are close to those that characterize c.1 and c.2, can be found in the terminal stage of the interception. In the terminal stage the missile and the target flights are characterized by Near Collision Course (*NCC*) conditions. Namely, the conditions are not precisely of a collision course, but they are close to them. In many cases of terminal stage collision course conditions are assumed in order to simplify the analysis. Therefore, a formulation of the motion described by c.1 and c.2 is important.

As the target and the missile velocities are constant, the relative velocity along the line of sight is also constant. Let constant relative velocity along the line of sight be the closing velocity v_c:

$$v_c = -\dot{\rho} \tag{2.1}$$

Where v_c is the constant closing velocity, ρ the length of the position vector r (*LOS* range).

Since the range between the missile and the target is supposed to be a decreasing function of time, v_c positive represents the constant rate in which the missile approaches the target.

To formulate collision-course conditions, let us substitute the ideal lead angle (1.8) to (1.4):

$$\begin{aligned}\rho \cdot \dot{\lambda} &= v_T \cdot sin\,\theta - v_m \cdot sin\left(sin^{-1}\left(\frac{v_T}{v_m} \cdot sin\,\theta\right)\right) \\ &= v_T \cdot sin\,\theta - v_m \cdot \frac{v_T}{v_m} \cdot sin\,\theta = 0\end{aligned} \tag{2.2}$$

Where λ is the line of sight angle and ρ the line of sight range, v_T and v_m are the target and the missile constant velocities, γ_T and γ_m their path angles, respectively. θ is target aspect angle and δ missile lead angle.

We get $\rho \cdot \dot{\lambda} = 0$, which implies that either $\rho = 0$ or $\dot{\lambda} = 0$. $\rho = 0$ is trivial, we seek solutions during pursuit time. Hence the *LOS* rate $\dot{\lambda}$ is zero. As a result the *LOS* angle λ is constant and the guidance command is zero $a_c = N \cdot v_m \cdot 0 = 0$. Second result is that the missile velocity direction is set constant (γ_m constant) and δ the lead angle and θ the aspect angle are also constant.

Now, uniform velocity of the target and the missile implies that the relative velocity along the line of sight is also constant:

$$\dot{\rho} = \underbrace{v_T \cdot cos(\gamma_T - \lambda)}_{const.} - \underbrace{v_m \cdot cos(\gamma_m - \lambda)}_{const.} = -v_c \tag{2.3}$$

Where $\dot{\rho}$ is the relative velocity along the line of sight, v_T and v_m are the target and the missile constant velocities, λ is the line of sight angle, γ_T and γ_m are the target and the missile path angles respective, v_c is the constant closing velocity.

Moreover, the time to collision may be calculated by the division of the range over the closing velocity. The prediction of the time-to-collision by a range measurement only is a significant simplification in the guidance design. Thus in some moment t_0 when collision-course conditions are satisfied, the range to the target is ρ_0, and the time interval remaining for the missile to fly until collision with the target is:

$$t_f = \rho_0 / v_c \tag{2.4}$$

Where t_f is the flight time from the measure of the range until collision, ρ_0 is the line of sight range at fixed moment t_0, v_c is the closing velocity.

Since the motion is linear, theoretically a single measure of ρ is enough to provide predictions for all the future times, because in later moments the time to collision may be found by subtracting from t_f the time that elapsed since the measure of ρ. This yields the variable t_{go}, time-to-go, that is used as an index for time-until-collision:

$$t_{go} = t_f - t \tag{2.5}$$

Where t_{go} is time-to-go, the current remained time until collision, t_f is the flight time from the measure of the range until collision and t is the time that elapsed since then. See Figure 2.2 for illustration.

Sometimes the term t_{go} is used even when linear motion is not the case. But it means the same and it is only an approximation to the remaining time-to-collision, based on the current range to target and closing velocity.

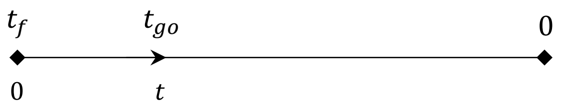

Figure 2.2: Timing illustration for time-to-go calculation. Labels below the line indicate a regular forward progressing timeline. Labels above indicate reverse order timeline starting at t_f.

2.2. Reference Trajectory

A state of operation around which the linearization should be performed can now be allocated.

During the linearization we assume ideal dynamics, since the dynamics model (1.16) is in anyway *LTI*, it has no effect on the results.

Let x_n be the state vector of the nominal trajectory that solves (1.20) and characterized by collision course conditions, then the variables of x_n are:

$$\begin{aligned} \rho_n &= v_c \cdot t_{go} \\ \lambda_n &= \lambda_0 \\ \omega_n &= 0 \\ \gamma_{T_n} &= \theta_n + \lambda_n \\ \gamma_{m_n} &= \delta_n + \lambda_n \end{aligned} \tag{2.6}$$

The nominal value of the variables $x_{a_n} = [\theta_n \quad \delta_n \quad a_c]$ is also derived from the ideal straight line motion:

$$\begin{aligned} \theta_n &= \theta_0 \\ \delta_n &= sin^{-1}(v_T/v_m \cdot sin\,\theta_n) = \delta_0, \qquad (\delta_{err} = 0) \\ a_c &= 0 \end{aligned} \tag{2.7}$$

Where subscript n stands for *nominal*, subscript 0 refers to initial value at time t_0, x_n is the state vector of the nominal trajectory, x_{a_n} is the vector of algebraic variables at the nominal trajectory, λ line of sight angle, ω line of sight angle rate, ρ line of sight range, v_c closing velocity, t_{go} time-to-go, γ_T and γ_m target and missile path angles, θ target aspect angle, δ missile lead angle, δ_{err} initial heading error, v_T and v_m target and missile velocities, a_c missile acceleration command.

On (2.6) and (2.7), target acceleration should also be appended, that from condition c.1 is $a_T = 0$. Notice that equations (1.20) and (1.21) do not include the input variable a_m, rather they have the input command a_c, that because during the linearization ideal dynamics assumption is valid, then x_d vanishes and $c_d \cdot x_d$ returns to be a_m again (see (1.16)).

The angles in (2.6) and (2.7) are in fact constants, determined by their initial values. However, δ_0 in the general case includes an error component, but for nominal motion the error δ_{err} is zero.

Recall that t_0 in (2.6) is a moment in which the missile motion satisfies near collision course conditions and was fixed as a reference for t_{go}, namely was tagged as t_f.

Now, a differentiation of (2.6) yields:

$$\dot{x}_n = [-v_c \quad 0 \quad 0 \quad 0 \quad 0]^T \tag{2.8}$$

Where subscript n stands for nominal, superscript T is the transpose operator, x_n nominal trajectory state vector, v_c closing velocity.

The nominal state (2.6) is confirmed as a solution by comparing the outcome (2.8) with the substitution of x_n and x_{a_n} from (2.6) and (2.7) to the *RHS* of (1.20).

2.3. Function Expansion

Let δx be a state vector of a deviation from the nominal trajectory x_n. The trajectory formed by a deviation upon the nominal trajectory is a general solution of (1.20):

$$x = x_n + \delta x \tag{2.9}$$

Where x is a general state vector that solves (1.20), x_n is state vector that solves (1.20) at the nominal trajectory and δx is a vector of variables at the environment of x_n.

The vector of variables x_{a_n} to calculate by the algebraic equations (1.21) at the nominal trajectory, is also the sum of a nominal component and a deviation component:

$$x_a = x_{a_n} + \delta x_a \tag{2.10}$$

Where x_a, x_{a_n} are the set of algebraic variables for the general solution and for the nominal trajectory respectively, δx_a is the vector of algebraic variables at the environment of x_{a_n}.

According to the linearization approach, the nonlinear variations have to be approximated by their parts, which are linear with respect to dt in Taylor series expansion:

$$\begin{gathered} \Delta x \cong \delta x \\ \Delta x_a \cong \delta x_a \end{gathered} \tag{2.11}$$

Where δx and δx_a are vectors at the environment of the nominal trajectory of the system state and of the algebraic-variables respectively, $\Delta x, \Delta x_a$ are linear approximation of $\delta x, \delta x_a$ respectively.

Differentiation of (2.9) yields $\dot{x} = \dot{x}_n + \delta\dot{x}$. Substitution of $\dot{x}_n + \delta\dot{x}$ at the left of (1.24) yields:

$$\dot{x}_n + \delta\dot{x} = f\left(x_n + \delta x, a_{T_n} + \delta a_T, a_{m_n} + \delta a_m\right) \tag{2.12}$$

Where x_n is a nominal trajectory state vector, δx is a state vector of a deviation, f is functions-vector *RHS* of (1.20), a_{T_n} and δa_T are target acceleration of nominal trajectory and of a deviation, a_{m_n} and δa_m are missile acceleration of nominal trajectory and of a deviation.

Function f in (2.12) can be expressed by a Taylor series with respect to x_n:

$$\begin{aligned} \dot{x}_n + \delta\dot{x} = f(z)|_{z=z_n} + \left.\frac{\partial f(z)}{\partial x}\right|_{z=z_n} \cdot \Delta x + \left.\frac{\partial f(z)}{\partial a_T}\right|_{z=z_n} \cdot \Delta a_T \\ + \left.\frac{\partial f(z)}{\partial a_m}\right|_{z=z_n} \cdot \Delta a_m + HOT \end{aligned} \tag{2.13}$$

Where z stands for the set of arguments $z = (x, a_T, a_m)$ and z_n stands for the nominal form of this set of arguments $z_n = (x_n, a_{T_n}, a_{m_n})$, $\frac{\partial}{\partial \varepsilon}$ is an operator of differentiation with respect to a variable ε (ε may be each of the arguments of z), f is the *RHS* of (1.20), x_n, a_{T_n}, a_{m_n} are the variables and inputs at the nominal trajectory, and $\Delta x, \Delta a_T, \Delta a_m$ are linear approximations of $\delta x, \delta a_T, \delta a_m$ – the state vector and inputs of a deviation from the nominal trajectory, acronym HOT stands for Higher Order Terms.

Taylor series (2.13), is an expansion of the set of functions f (1.24) (1.20) about the nominal trajectory (2.6). In (2.13), only the first derivative (for each one of the arguments) appears explicitly, higher derivatives are given implicitly by HOT (Higher Order Terms).

Since we seek linear approximation to δx, terms higher than the first derivative are not of interest, therefore HOT will be omitted. In addition, the term $f(z)|_{z=z_n}$, i.e. $f(x, a_T, a_m)|_{x,a_T,a_m=x_n,a_{T_n},a_{m_n}}$, in (2.13), is in fact $\dot{x}_n$. Considering these and (2.11), (2.13) becomes:

$$\begin{aligned}\Delta\dot{x} = & \left.\frac{\partial f(x,a_T,a_m)}{\partial x}\right|_{x,a_T,a_m=x_n,a_{T_n},a_{m_n}} \cdot \Delta x \\ & + \left.\frac{\partial f(x,a_T,a_m)}{\partial a_T}\right|_{x,a_T,a_m=x_n,a_{T_n},a_{m_n}} \cdot \Delta a_T \\ & + \left.\frac{\partial f(x,a_T,a_m)}{\partial a_m}\right|_{x,a_T,a_m=x_n,a_{T_n},a_{m_n}} \cdot \Delta a_m\end{aligned} \tag{2.14}$$

Where $\Delta\dot{x}$ is linear approximation of $\delta\dot{x}$, the time-derivative of a deviation from the nominal trajectory x_n, $\frac{\partial}{\partial\varepsilon}$ is an operator of differentiation with respect to a variable ε *(*$\varepsilon = x$ or a_T or a_m*)*, $f(x, a_T, a_m)$ is *RHS* of (1.20), x_n, a_{T_n}, a_{m_n} are the state vector and inputs of the nominal trajectory, and $\Delta x, \Delta a_T, \Delta a_m$ are linear approximation of $\delta x, \delta a_T, \delta a_m$ – the state vector and inputs of a deviation from the nominal trajectory.

$\frac{\partial f(x,a_T,a_m)}{\partial x}$ in (2.14) is first-order partial derivative of the functions-vector f (1.24) of (1.20), with respect to each variable in the state vector x. After the differentiations, the variables in the expressions have to be replaced by their nominal values. By the same manner $\frac{\partial f(x,a_T,a_m)}{\partial a_T}$ and $\frac{\partial f(x,a_T,a_m)}{\partial a_m}$ are first-order partial derivatives of the functions-vector f with respect to a_T and a_m respectively.

Function that involves θ or δ has to be substituted first with the *RHS* of the variable. For example, the first function in (1.20) is $\dot{\rho} = f_1(x, a_T, a_m) = v_T \cdot cos\,\theta - v_m \cdot cos\,\delta$, its partial derivative with respect to $x_2 = \lambda$ will therefore be:

$$\begin{aligned}\frac{\partial f_1}{\partial \lambda} &= \frac{\partial[v_T \cdot cos(\gamma_T - \lambda) - v_m \cdot cos(\gamma_m - \lambda)]}{\partial \lambda} \\ &= v_T \cdot sin(\gamma_T - \lambda) - v_m \cdot sin(\gamma_m - \lambda)\end{aligned} \tag{2.15}$$

Where $\frac{\partial}{\partial \varepsilon}$ is an operator of differentiation with respect to a variable ε, $f_1 = \dot{\rho}$, λ is the *LOS* angle, v_T and v_m target and missile velocities, γ_T and γ_m are their angles respectively.

Substitution of the nominal variables, in this case only the nominal algebraic variables $\theta_n = \theta_0$ and $\delta_n = \delta_0$, provides the coefficient of Δx_2 for the linearized function of $\dot{x}_1$:

$$\left.\frac{\partial f_1}{\partial x_2}\right|_{x_a = x_{a_n}} = v_T \cdot sin\,\theta_0 - v_m \cdot sin\,\delta_0$$
$$= v_T \cdot sin\,\theta_0 - v_m \cdot sin\left[sin^{-1}\left(sin(\theta_0) \cdot \frac{v_T}{v_m}\right)\right] = 0 \tag{2.16}$$

Where $\frac{\partial f_1}{\partial x_2}$ is differentiation of f_1 with respect to x_2, $f_1 = \dot{\rho}$ is the first function in the *RHS* of (1.20), x_a is the vector of the dependent variables of algebraic equations (1.21), x_{a_n} is the vector x_a at the nominal trajectory, v_T and v_m target and missile velocities, θ_0 is the target aspect angle at time t_0 and δ_0 is the missile lead angle at time t_0.

The coefficient of Δx_2 in the linearized function f_1 happens to be zero. Similar process, when applied to the rest of the coefficients of f_1 and to all the other functions, yields the final form of (2.14), which by including the terms of the missile dynamics is given by the following set of linearized differential equations:

$$\begin{aligned}
\Delta\dot{x}_1 &= -v_T \cdot sin\,\theta_n \cdot \Delta x_4 + v_T \cdot sin\,\theta_n \cdot \Delta x_5 \\
\Delta\dot{x}_2 &= \Delta x_3 \\
\Delta\dot{x}_3 &= \frac{2}{t_{go}} \cdot \Delta x_3 + \frac{c_T}{t_{go}} \cdot \Delta a_T - \frac{c_m}{t_{go}} \cdot c_d \cdot \Delta x_d \\
\Delta\dot{x}_4 &= \frac{1}{v_T} \cdot \Delta a_T \\
\Delta\dot{x}_5 &= \frac{1}{v_m} \cdot c_d \cdot \Delta x_d \\
\Delta\dot{x}_d &= A_d \cdot \Delta x_d + b_d \cdot \Delta a_c
\end{aligned} \tag{2.17}$$

Where subscript n stands for nominal, subscript d stands for dynamics, $[\Delta x_1, \ldots, \Delta x_5] = [\Delta\rho, \Delta\lambda, \Delta\omega, \Delta\gamma_T, \Delta\gamma_m]$ are the linearized state variables, $\Delta x_d \in R^{n\times 1}$ is the state vector of the missile dynamics with state matrix $A_d \in R^{n\times n}$ and vectors $b_d \in R^{n\times 1}$, $c_d \in R^{1\times n}$, Δa_c and Δa_T are inputs of the missile acceleration and the target acceleration respectively, $c_T = cos\,\theta_n / v_c$ and $c_m = cos\,\delta_n / v_c$ are constant coefficients, t_{go} is the remaining time to collision (time-to-go), v_c is the constant closing velocity, v_T and v_m target and missile constant velocity magnitudes, θ_n initial target aspect angle and δ_n initial ideal missile lead angle.

The constant coefficients c_T, c_m are given by:

$$c_T = \frac{cos\,\theta_n}{v_c}$$
$$c_m = \frac{cos\,\delta_n}{v_c} \quad (2.18)$$

Where c_T is the constant coefficient of Δa_T and c_m is the constant coefficient of Δa_c with the missile dynamics, $\theta_n = \theta_0$ is the target initial aspect angle and δ_n is the missile ideal lead angle, v_c is the closing velocity.

Linearization of vector x_a of the algebraic variables yields:

$$\Delta\theta = \Delta\gamma_T - \Delta\lambda$$
$$\Delta\delta = \Delta\gamma_m - \Delta\lambda \quad (2.19)$$
$$\Delta a_c = N \cdot v_m \cdot \Delta\omega$$

Where $\Delta\theta$ is the target linearized aspect angle and $\Delta\delta$ missile linearized lead angle, $\Delta\gamma_T$ and $\Delta\gamma_m$ are target and missile linearized path angles respectively, $\Delta\lambda$ linearized *LOS* angle, Δa_c is linearized acceleration command needed to perform by the control system, N constant navigation gain, v_m missile velocity and $\Delta\omega$ the linearized line of sight angle rate.

Where $\Delta x_a = x_a$ remained with no change because the already linear form of the equations. In practice $\Delta\theta$ and $\Delta\delta$ need not be calculated. These variables are replaced by their nominal constant values θ_n and δ_n. θ_n is the initial target aspect angle, and δ_n is the initial missile ideal lead angle, with no error.

The terms t_{go} and v_c in (2.17) and (2.18) are products of the linear motion about which the linearization is done. As shown in section 2.1, in collision-course conditions the trajectories of the target and the missile are straight lines that are about to meet, where the time to collision is a function of the instantaneous range and the constant closing velocity.

The terms to find the parameters at the right hand side of (2.17) are collected from previous equations:

$$\theta_n = \theta_0$$
$$\delta_n = sin^{-1}(sin(\theta_n) \cdot v_T/v_m) = \delta_0 - \delta_{err}$$
$$v_c = v_m \cdot cos\,\delta_n - v_T \cdot cos\,\theta_n$$
$$c_T = cos\,\theta_n\,/v_c \quad (2.20)$$
$$c_m = cos\,\delta_n\,/v_c$$
$$t_f = \rho_0/v_c$$
$$t_{go} = t_f - t$$

Where subscript n stands for nominal, subscript 0 denotes initial time t_0, θ target aspect angle and δ missile lead angle, δ_{err} initial heading error, v_m missile constant velocity magnitude, v_T target constant velocity magnitude, v_c closing velocity, c_T and c_m are constant coefficients, t_{go} is time-to-go, t_f is the flight time, t is the time that elapsed since the initial time t_0, ρ line of sight range.

From (2.9), initial conditions for (2.17) are found by calculating the difference of the general solution and the nominal trajectory (2.6):

$$\begin{aligned} x_{1_0} &= 0 \\ x_{2_0} &= 0 \\ x_{3_0} &= (v_T \cdot sin\,\theta_0 - v_m \cdot sin\,\delta_0)/(v_c \cdot t_f) \\ x_{4_0} &= 0 \\ x_{5_0} &= \delta_{err} \\ x_{d_0} &= 0 \end{aligned} \tag{2.21}$$

Where $x_0 \in R^{(n+5)\times 1}$ is the initial conditions vector of the linearized system (2.17), ρ_0 line of sight range at time t_0, v_T and v_m target and missile velocities, θ_0 is the target aspect angle at time t_0 and δ_0 is the missile lead angle at time t_0, δ_{err} initial heading error – the deviation of the missile lead angle from the correct direction (see 1.9), x_d state vector of missile dynamics of n^{th} order.

2.4. Linear System – Examples

The systems that will be used to demonstrate the linearization appear in the following equations. As described in section 2.3, approximation of the nonlinear state is composed of two parts, nominal part and a linearized deviation. Therefore for each system here, the solution of the nominal reference is given as well as the differential equations of the linearized deviation.

Parameters and conditions of the nonlinear examples 1.5 still hold.

Ideal Dynamics

Algebraic calculations for the reference and the linear system:

$$\begin{aligned} \theta_n &= \theta_0 \\ \delta_n &= \delta_0 - \delta_{err} \\ \lambda_n &= \lambda_0 \\ v_c &= 400 \cdot cos\,\delta_n - 200 \cdot cos\,\theta_n \\ c_T &= cos\,\theta_n\,/v_c \\ c_m &= cos\,\delta_n\,/v_c \\ t_f &= \rho_0/v_c \end{aligned} \tag{2.22}$$

$$t_{go} = t_f - t$$

Supporting calculations:

$$\begin{aligned}
&\rho_0 = x_{T0} \\
&\lambda_0 = sin^{-1}(r_{y_0}/\rho) = 0 \\
&\omega_0 = (200 \cdot sin\,\theta_0 - 400 \cdot sin\,\delta_0)/x_{T0} \\
&\gamma_{T_0} = \theta_0 \\
&\gamma_{m_0} = sin^{-1}[(sin\,\theta_0)/2] + \delta_{err}
\end{aligned} \tag{2.23}$$

Reference System

System equations:

$$\begin{aligned}
&x_1 = v_c \cdot t_{go} \\
&x_2 = \lambda_n \\
&x_3 = 0 \\
&x_4 = \theta_n + \lambda_n \\
&x_5 = \delta_n + \lambda_n
\end{aligned} \tag{2.24}$$

Linear System

Differential equations:

$$\begin{aligned}
&\dot{x}_1 = -200 \cdot sin\,\theta_n \cdot x_4 + 200 \cdot sin\,\theta_n \cdot x_5 \\
&\dot{x}_2 = x_3 \\
&\dot{x}_3 = 2 \cdot x_3/t_{go} - c_m \cdot a_c/t_{go} \\
&\dot{x}_4 = 0 \\
&\dot{x}_5 = a_c/400
\end{aligned} \tag{2.25}$$

Initial conditions:

$$\begin{aligned}
&x_{1_0} = 0 \\
&x_{2_0} = 0 \\
&x_{3_0} = (200 \cdot sin\,\theta_0 - 400 \cdot sin\,\delta_0)/(v_c \cdot t_f) \\
&x_{4_0} = 0 \\
&x_{5_0} = \delta_{err}
\end{aligned} \tag{2.26}$$

1st Order Dynamics

Based on the first order transfer function:

$$a_m = \frac{1}{\tau_m \cdot s + 1} \cdot a_c \tag{2.27}$$

Where a_m is the missile actual acceleration at the lateral plane, a_c PN command, s Laplace transform variable and τ_m time-constant of the missile dynamics.

Reference System

System equations:

$$\begin{aligned} x_1 &= v_c \cdot t_{go} \\ x_2 &= \lambda_n \\ x_3 &= 0 \\ x_4 &= \theta_n + \lambda_n \\ x_5 &= \delta_n + \lambda_n \\ x_6 &= 0 \end{aligned} \tag{2.28}$$

Linear System

Differential equations:

$$\begin{aligned} \dot{x}_1 &= -200 \cdot sin\,\theta_n \cdot x_4 + 200 \cdot sin\,\theta_n \cdot x_5 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= 2 \cdot x_3/t_{go} - c_m \cdot x_6/t_{go} \\ \dot{x}_4 &= 0 \\ \dot{x}_5 &= x_6/400 \\ \dot{x}_6 &= -x_6/\tau_m + a_c/\tau_m \end{aligned} \tag{2.29}$$

Initial conditions:

$$\begin{aligned} x_{1_0} &= 0 \\ x_{2_0} &= 0 \\ x_{3_0} &= (200 \cdot sin\,\theta_o - 400 \cdot sin\,\delta_0)/(v_c \cdot t_f) \\ x_{4_0} &= 0 \\ x_{5_0} &= \delta_{err} \\ x_{6_0} &= a_{m_0} \end{aligned} \tag{2.30}$$

Example 1 – States Approximation

The purpose now is to examine the approximation of the linear states, most importantly $\rho = x_1$ and $\omega = x_3$, to their nonlinear values. The line of sight rate ω according to the proportional navigation law, is the feedback control state and its asymptotic behavior is a key factor for stability in the sense of Lyapunov. ρ is the missile-target relative position and its final value determines the miss distance. Without reliable approximation to these two, no significant results can be achieved.

The simulation in the following examples ran the linear and nonlinear systems for ideal and nonideal dynamics, for initial heading error δ_{err_0}.

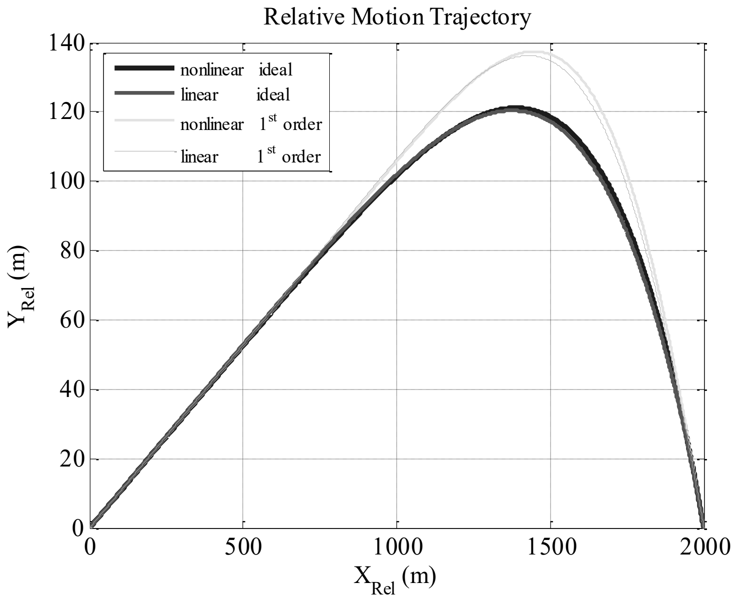

Figure 2.3: Relative motion trajectories comparison of linear and nonlinear systems, for ideal system and for first order dynamics.

The linearized states of the relative position (Figure 2.3) and the line of sight rate (Figure 2.4) present very good approximation to their nonlinear values. In both figures a gap distinguishes the nonideal (1^{st} order) system from the ideal system. But the signals of the linear system follow closely in time (Figure 2.4) and in position (Figure 2.3) to their original signals.

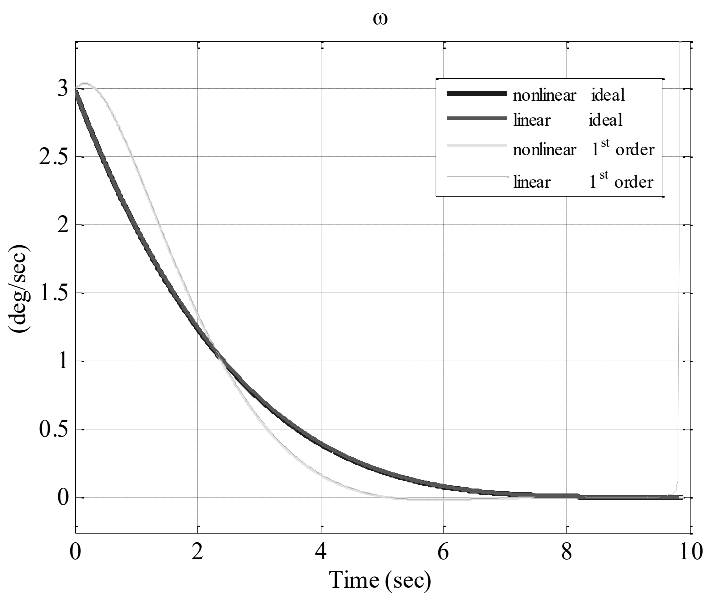

Figure 2.4: Comparison of acceleration histories of linear and nonlinear systems, for ideal system and for first order dynamics.

Example 2 – Miss Distance

Now we will study the miss-distance estimations as provided by the linear system. Figure 2.5 displays the distance between the missile and target at the time of flyby, the time at which the closing velocity changes direction from approaching to distancing, versus initial heading error δ_{err_0}. In order to achieve distinct results, the comparison has been done using a nonideal dynamic system, in this case 1st order dynamics.

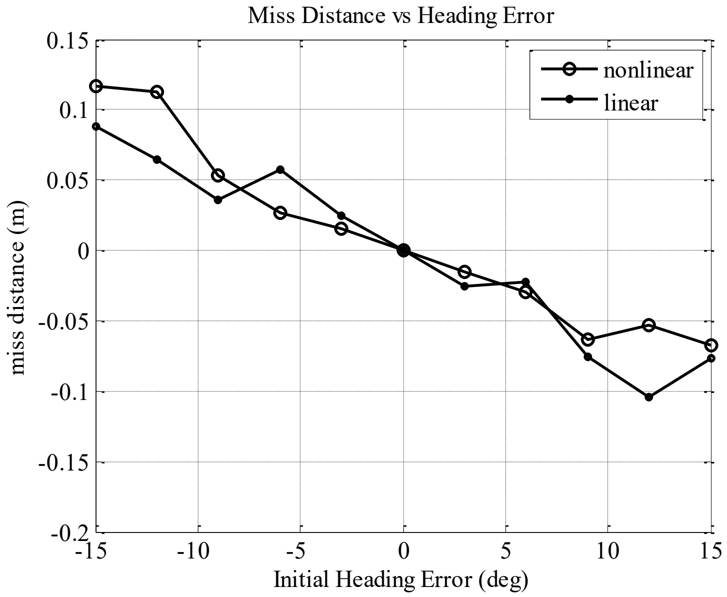

Figure 2.5: Miss Distance comparison.

The miss distance achieved by the linear system reflects the original miss distance of the nonlinear system very well. Recall the state of operation about which the linearization was done is ideal, namely $\delta_{err_0} = 0$, so it is obvious therefore that the better approximation will be at the vicinity of that condition.

In the next chapter the linearized model will be the basis for the stability analysis of a guidance system engaging proportional navigation.

2.5. Linear Model Overview

The relation of the missile and the target accelerations to the line of sight rate that was a concern in section 1.4.1 and indicated by the operator $T(t)$, is now termed explicitly with time-variable coefficients, as indicated in the third equation in (2.17):

$$\Delta\dot{x}_3 = \frac{2}{t_{go}} \cdot \Delta x_3 + \frac{c_T}{t_{go}} \cdot \Delta a_T - \frac{c_m}{t_{go}} \cdot c_d \cdot \Delta x_d \tag{2.17}$$

Where $[\Delta x_1, \dots, \Delta x_5] = [\Delta\rho, \Delta\lambda, \Delta\omega, \Delta\gamma_T, \Delta\gamma_m]$ are the linearized state variables, $\Delta x_d \in R^{n\times 1}$ is the state vector of the missile dynamics with state matrix $A_d \in R^{n\times n}$ and vectors $b_d \in R^{n\times 1}$, $c_d \in R^{1\times n}$, Δa_T is the input of the target acceleration, $c_T = \cos\theta_n / v_c$ and $c_m = \cos\delta_n / v_m$ are constant coefficients, t_{go} is the remaining time to collision (time-to-go), v_c constant closing velocity.

The leading term $t_{go} = t_f - t$ is the time-to-go, the remaining time to collision starting with fixed point in time t_f (see Figure 2.2).

Now, the linearized model (2.17) is presented schematically in Figure 2.6. In it we identify three subsystems (each encompassed with a dashed line).

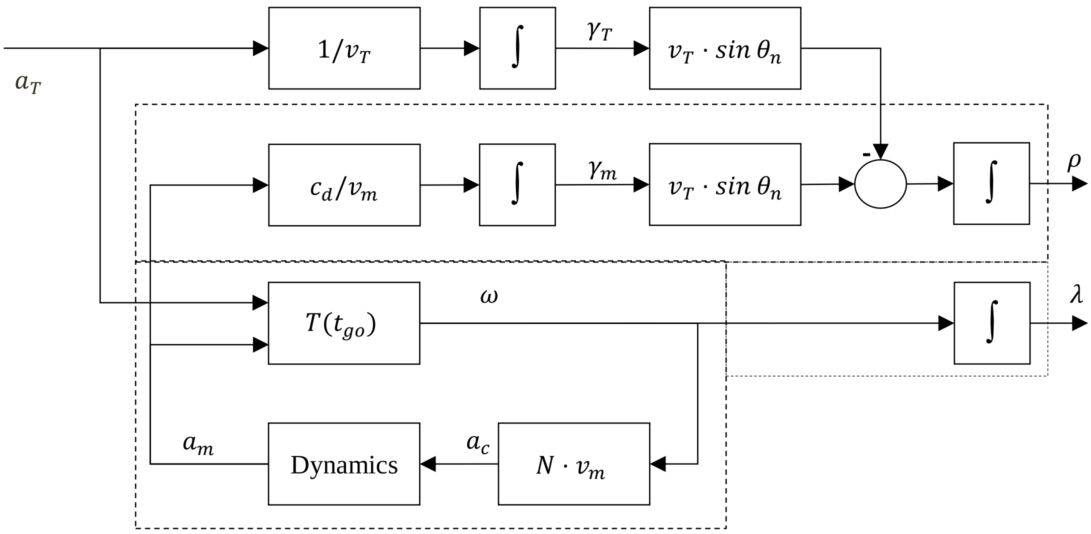

Figure 2.6: Linearized model description with dashed lines mark the different subsystems.

The first subsystem appearing in Figure 2.6 is of the operator $T(t)$. It is a closed loop about $x_3 = \omega$ with ω at the output. This subsystem is governed by equations (3) and (6) in (2.17). We term this part the ‘closed-loop’ system.

The second subsystem begins with a_c and ends with $x_2 = \lambda$. This system contains a single integrator after the closed-loop (Recall that a_c is the proportional navigation command $a_c = N \cdot v_m \cdot \omega$). In (2.17), equation (2) forms this relation.

The third is the subsystem that begins with a_c and ends with $x_1 = \rho$. The behavior of this system is the behavior of the missile-target range. After the closed-loop system, two integrators appear in this subsystem. Contained in equations (1) and (5) in (2.17).

Notice that the integrations of the second and the third subsystems cannot be separated or deferred from the integration of the closed-loop system. That is, the overall system (2.17) has to be integrated simultaneously.

The closed-loop system connects the missile acceleration command with the line of sight angle rate derivative and it possesses the main stability properties of the system. Chapter 3 will be devoted to the analysis of that subsystem.

As the final value of ρ forms the miss-distance, the third subsystem has significance in miss distance analysis. But as this system has two integrators after the closed-loop system and the second subsystem has only one, the stability of this subsystem cannot be examined separately of the second. Chapter 4 will introduce the analysis of these two subsystems.

2.6. Summary

Chapter 2 developed the linearized model of the guidance system. It has been demonstrated that collision-course conditions, which are an ideal state of the proportional navigation system, can serve as basis for linearization. A linearized model in the form of ordinary linear differential equations with time-varying parameters was derived. Examples for time histories of the states and miss-distance results, demonstrated very good approximation with the exact nonlinear model.

3. Closed-Loop Analysis

The models for both nonlinear and linearized systems were developed in the previous chapters. We now move on to analyze the stability of the system. In the following chapter an analysis of the line of sight angle rate is given. This variable requires particular consideration because of the special shape of its time-dependency. In the opening, an introduction about the problem, later on is model rearrangement and finally analysis based on an extension to Lyapunov stability theory.

3.1. The Stability Problem

3.1.1. The Linear Model

For the purpose of the discussion in this chapter let's rewrite the final form of the linearized guidance system developed in chapter 2, in terms of a state-space model:

$$\begin{aligned}
\dot{x}_1 &= -v_T \cdot sin\,\theta_n \cdot x_4 + v_T \cdot sin\,\theta_n \cdot x_5 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= \frac{2}{t_{go}} \cdot x_3 + \frac{c_T}{t_{go}} \cdot a_T - \frac{c_m}{t_{go}} \cdot c_d \cdot x_d \\
\dot{x}_4 &= \frac{1}{v_T} \cdot a_T \\
\dot{x}_5 &= \frac{1}{v_m} \cdot c_d \cdot x_d \\
\dot{x}_d &= A_d \cdot x_d + b_d \cdot a_c
\end{aligned} \tag{3.1}$$

Where subscript n stands for nominal, subscript d stands for dynamics, $[x_1, \dots, x_5] = [\rho, \lambda, \omega, \gamma_T, \gamma_m]$ are the state variables, $x_d \in R^{n\times 1}$ is the state vector of the missile dynamics with state matrix $A_d \in R^{n\times n}$ and vectors $b_d \in R^{n\times 1}$, $c_d \in R^{1\times n}$, a_c and a_T are inputs of the missile acceleration and the target acceleration respectively, $c_T = cos\,\theta_n / v_c$ and $c_m = cos\,\delta_n / v_c$ are constant coefficients, t_{go} is the remaining time to collision (time-to-go), v_c is the constant closing velocity, v_T and v_m target and missile constant velocity magnitudes, θ_n initial target aspect angle and δ_n initial ideal missile lead angle.

Solving equations (3.1) is subject to a preceding evaluation of the algebraic equations and the following expressions:

$$\begin{aligned}
a_c &= N \cdot v_m \cdot \omega \\
\theta_n &= \theta_0 \\
\delta_n &= sin^{-1}(sin(\theta_n) \cdot v_T / v_m) = \delta_0 - \delta_{err}
\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
& v_c = v_m \cdot cos\, \delta_n - v_T \cdot cos\, \theta_n \\
& c_T = cos\, \theta_n \,/ v_c \\
& c_m = cos\, \delta_n \,/ v_c \\
& t_f = \rho_0 / v_c \\
& t_{go} = t_f - t
\end{aligned}
$$

Where subscript n stands for nominal, subscript 0 denotes initial time t_0, a_c is the acceleration command needed to perform by the control system, N constant navigation gain, v_m missile velocity and ω the line of sight angle rate, θ target aspect angle and δ missile lead angle, δ_{err} initial heading error, v_T target constant velocity magnitude, v_c closing velocity, c_T and c_m are constant coefficients, t_{go} is the time-to-go, t_f is the flight time, t is the time that elapsed since the initial time t_0, ρ line of sight range.

Initial conditions for (3.1) are given by:

$$
\begin{aligned}
& x_{1_0} = 0 \\
& x_{2_0} = 0 \\
& x_{3_0} = (v_T \cdot sin\, \theta_0 - v_m \cdot sin\, \delta_0)/(v_c \cdot t_f) \\
& x_{4_0} = 0 \\
& x_{5_0} = \delta_{err} \\
& x_{d_0} = 0
\end{aligned}
\tag{3.3}
$$

Where $x_0 \in R^{(n+5)\times 1}$ is the initial conditions vector of system (3.1), v_T and v_m target and missile velocities, θ_0 is the target aspect angle at time t_0 and δ_0 is the missile lead angle at time t_0, δ_{err} initial heading error – the deviation of the missile lead angle from the correct direction (see 1.9), x_d state vector of missile dynamics of n^{th} order, v_c is the constant closing velocity and t_f is the flight time.

A note about initial conditions: According to Lyapunov, we do not take into account the initial state (3.3) of the system. The theory allows assuming any initial state $x_0 \in R^{(n+5)\times 1}$ taken arbitrarily in a small neighborhood of the origin (in the linear case, the vicinity of zero has no limits).

3.1.2. Stability Analysis of Time-Varying Systems

The concept of Lyapunov-stability for time-varying systems is an extension to the fundamental Lyapunov theory. Like the fundamental case, stability of a time-varying system defined around an equilibrium point. If the initial conditions may be selected independently of t_0, the system is said to be uniformly stable in the sense of Lyapunov.

Lyapunov function for time-varying systems usually has to include a time variable within it and then the time-derivative of the function consists of explicit terms of t:

$$\dot{v}(x(t),t) = \frac{\partial v(x,t)}{\partial t} + \nabla v(x,t) \cdot f(x(t),t) \tag{3.4}$$

Where v is the Lyapunov function, $x(t)$ is the state vector, $\frac{\partial}{\partial t}$ is the time derivative operator, ∇ is the gradient with respect to the state variables of $x(t)$, f is a functions-vector representing the derivatives of $x(t)$, t time variable.

$v(x,t)$ is a Lyapunov function of a system around an equilibrium point, if it is positive definite on $t \geq 0$ and $\dot{v}(x(t),t) \leq 0$. As it is often difficult to find a Lyapunov function for a system, in the case of a guidance system it is actually impossible, since the function candidate to be Lyapunov and its first order partial derivatives, have to be defined and continuous over the all interval $t \geq 0$, a condition that cannot be applicable to (3.1) because the system has a singular point at $t = t_f$ and is not defined for later times.

3.1.3. Sectorial Method Analyses

Previous works studied the stability of proportional navigation systems using sectorial methods such as the Popov criterion [8] and the Circle criterion [11]. In these methods the nonlinear part of the system, denoted by a continuous function $\varphi(y)$, where y feedback state vector, belongs to a sector $[k_1, k_2]$ [12]:

$$y \neq 0 \implies k_1 \leq \frac{\varphi(y)}{y} \leq k_2 \tag{3.5}$$

By assigning $\varphi(y)$ with the time-varying part of the linearized guidance model, the sector of time in which the 'system remains finite-time stable' is found. As a result, sufficient conditions to stability are derived:

$$\begin{gathered} N > 2 \\ t_{go} > -N \cdot R[G(i\omega)/i\omega] \end{gathered} \tag{3.6}$$

Where R implies real part, N is the navigation gain, t_{go} time-to-go, $G(i\omega)$ missile subsystems transfer function, ω frequency domain variable, i is the imaginary number.

According to (3.6), for a missile represented by a 1^{st} order transfer function with time-constant $\tau_m = 0.5\ sec$ and gain $N = 4$, the system remains stable for every $t_{go} < -4 \cdot (-0.5) = 2\ sec.$, but no longer.

Similar results that provide conditions to stability in terms of the smallest stable time t_{go}, or smallest stable range ρ, generated by a customary approach of engineers to freeze the system in different times along the process and to apply standard tools as if the system were an LTI [2].

3.1.4. Differential Equations with Terminal Singular Point

The stability question is sometimes confused with the question of successful interception. The problem the designer of a missile faces is to hit a target in a given range within a given time. And if the system is stable, then the line of sight rate will remain bounded and the missile should meet the target. Then maybe it is a matter of some given parameters, that the stability has to be examined with respect to them. In this case the flight time is fixed and there is no sense in talking about asymptotic stability.

But if the question of stability is examined in the broad sense, then it has to query the guidance system character, as to whether it is stable or not. Namely, for the general interception problem of a guidance system, represented in a parametric manner, whether it is stable or not. In that sense, meaning to asymptotic stability exists, because if the state vector norm decreases with respect to an increase of the target range, which is equivalent to an increase in flight time, then the system is asymptotic-stable for sure. The problem for each individual interception then will be to find the appropriate conditions for launching. In this regard then, stability does finally provide an answer to the question of performances.

System (3.1) is time-varying. But the varying coefficients of (3.1) are not simple factors of time, rather they depend inversely on time-to-go $(1/t_{go} = 1/(t_f - t))$, which implies that the process is limited within a finite time t_f (see Figure 2.2), and as time elapses, and t_{go} approaches 0, the time variables go to infinity.

The presence of t_{go} in the dominator creates a terminal (final-time) singular point in the system model. Mathematically it makes the set of equations (3.1) a complicated form of ordinary differential equations and the orthodox approaches to investigate time-varying systems are therefore not applicable in this case. In fact only a theory of ordinary linear differential equations with terminal singular point, may provide results that do justice to the unique form of the system.

Accordingly we seek to analyze the stability of the guidance loop of a missile employing proportional navigation by querying the linearized model (3.1), and to do so while taking into account the characters of the model, which is time-varying and has a terminal singular point.

The problem of the stability here is to find the guidance system parameters that guarantee that the state vector of (3.1) remains bounded, or better, asymptotically converges, over the flight.

However, as the terminal singular point of the model indicates, and is evident by the geometry of the pursuit, as the missile approaches the target, the line of sight angle-rate tends to diverge. Then a final divergence is a natural result of the guidance system.

Regarding that, an additional effort has to be made in order to consider the final moment in the analysis. In the next section an approach is presented that meets the requirements to analyze system (3.1).

3.2. Barabanov - Skorokhod Approach

An approach to the theory of linear systems with a regular singular point at the end of the process was introduced by Barabanov and Skorokhod [13] [14]. In their papers the analyzed system has the following mathematical model:

$$\begin{aligned} \dot{x} &= F \cdot x + g \cdot \frac{y}{t_{go}} \\ y &= h \cdot x \end{aligned} \tag{3.7}$$

Where $\dot{x}, x \in R^{n\times 1}$ are the analyzed system state vector and its derivative, $F \in R^{n\times n}, g \in R^{n\times 1},\ h \in R^{1\times n}$ are the system state matrix and vectors, y is a state feedback control input, $t_{go} = t_f - t$ is the remained time to end of the process (time-to-go. t_f total process time, t time variable).

State matrix F and vectors g and h in (3.7) are constant and F has left half-plane eigenvalues, with a possible exception for one zero.

The analysis of systems such as (3.7), according to the Barabanov - Skorokhod approach, is performed by exploring the behavior of the system for non-bounded increase of the process time-length ($t_f \to \infty$), using an extension to Lyapunov theorem.

Based on this concept, Definition 1 and Definition 2 below were introduced by Barabanov [13]. The definitions make use of the following symbols:

$x(t)$	Time-variable state vector of the analyzed system
x_0	Initial conditions of state vector $x(t)$

$\|x\|$	Euclidean norm of vector x
t_f	Total flight time of a single process
t_{go}	Time-to-go, remained time to collision
α	Positive constant
τ_1	Positive constant
τ_2	Positive constant

Definition 1

System (3.7) is uniformly stable (uniformly bounded) if there exists a positive constant $\alpha > 0$, such that the inequality

$$\|x(t)\| \leq \alpha \cdot \|x_0\| \tag{3.8}$$

holds for every $0 \leq t \leq t_f - \tau_1$, where τ_1 is an arbitrary positive constant.

Definition 2

System (3.7) is asymptotically stable, if it is uniformly stable and for any two positive constants, $0 < \tau_1 < \tau_2$, the system state $\|x(t)\|$ approaches zero as $t_f \to \infty$, where:

$$\tau_1 \leq t_{go} \leq \tau_2 \tag{3.9}$$

Notes on Definition 1 and Definition 2:

n.1. The time parameters and chronological order in Definition 1 and Definition 2 may be elucidated with Figure 3.1:

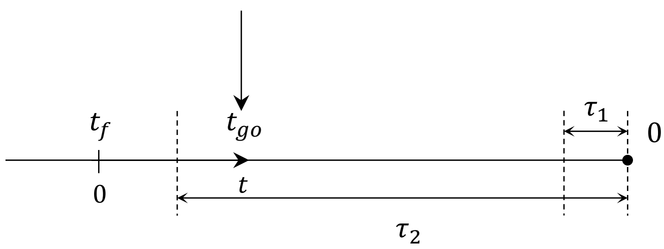

Figure 3.1: Timing illustration for Definition 1 and Definition 2. The labels below the line indicate a regular forward progressing timeline; the labels above indicate a reverse order timeline.

n.2. The arbitrary constant $\alpha > 0$ does not depend on t_f.

n.3. The arbitrary positive constant τ_1 was introduced in Definition 1 to avoid the analysis of the system in the neighborhood of the singular point $t_{go} = 0$. By that gap, all the statements about the system exclude the final moment, where the missile guidance is practically uncontrollable. See Figure 3.1 for illustration.

n.4. The free variable that progresses asymptotically here is t_f rather than t. Since the guidance process holds between t_f and 0, for each process t is bounded over that interval of time. Moreover, Definition 2 states that a missile is regarded asymptotically stable, on condition that on time interval (3.9), vector norm $\|x\|$ decreases for an increment of t_f.

n.5. Thus, the Barabanov - Skorokhod approach to stability of the closed-loop system in the sense of Lyapunov, means that its state is bounded (tends toward zero) for a non-bounded increase of the flight time ($t_f \to \infty$).

In the next section we are going to organize the system states to correspond with the mathematical form of model (3.7).

3.3. Model Rearrangement

In order to bring system (3.1) to the appropriate form as required by the Barabanov - Skorokhod approach, a linear transformation has to be performed. But first, we seek to simplify the system by reducing its order, as suggested in the form of a number of equations in (3.1).

3.3.1. Order Reduction

First let's assume a non-maneuvering target. Since stability in the sense of Lyapunov is unaffected by the target acceleration, we henceforth put [7]:

$$a_T = 0 \tag{3.10}$$

Where a_T is target acceleration input.

This allows us to omit $\dot{x}_4$ in (3.1) and replace x_4 with $x_{4_0} = 0$.

Recall also that the guidance acceleration command is given by the *PN* law (1.14):

$$a_c = N \cdot v_m \cdot \omega \tag{1.14}$$

Where a_c is a lateral acceleration command needed to perform by the control system, N constant navigation gain, v_m missile velocity and ω the line of sight angle rate.

And let (1.14) replace a_c in the equations.

Regarding these, (3.1) becomes:

$$\begin{aligned}
\dot{x}_1 &= v_T \cdot \sin\theta_n \cdot x_5 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= \frac{2}{t_{go}} \cdot x_3 - \frac{c_m}{t_{go}} \cdot c_d \cdot x_d \\
\dot{x}_5 &= \frac{1}{v_m} \cdot c_d \cdot x_d \\
\dot{x}_d &= A_d \cdot x_d + b_d \cdot a_c
\end{aligned} \tag{3.11}$$

Where subscript n stands for nominal, subscript d stands for dynamics, $[x_1, x_2, x_3, x_5] = [\rho, \lambda, \omega, \gamma_m]$ are the state variables, $x_d \in R^{n\times1}$ is the state vector of the missile dynamics with state matrix $A_d \in R^{n\times n}$ and vectors $b_d \in R^{n\times1}$, $c_d \in R^{1\times n}$, a_c input of the missile acceleration, $c_m = \cos\delta_n / v_c$ is a constant coefficient, t_{go} is the remaining time to collision (time-to-go), v_c is the constant closing velocity, v_m missile constant velocity magnitude, θ_n initial target aspect angle and δ_n initial ideal missile lead angle.

Now, as shown in section 2.5 and in Figure 2.6, three different subsystems can be distinguished in (3.11). With the internal numbering of the equations in (3.11) the subsystems are: one, the closed-loop system, governed by equations (3) and (5); two, a subsystem that begins with $x_3 =$ ω and ends with $x_2 = \lambda$, equation (2) forms this relation; three, a subsystem that begins with a_m and ends with $x_1 = \rho$, in equations (1) and (4).

The closed-loop system connects the missile acceleration command with the line of sight angle-rate derivative and it possesses the main stability properties of the system. Therefore, this part is of our interest in the current chapter.

The next chapter will be devoted to the analysis of the two other subsystems.

Regarding all that, the final form to serve as the basis for the transformation in the next section is summed-up by the following:

$$\begin{aligned}
\dot{x} &= \frac{2}{t_{go}} \cdot x - \frac{\cos\delta_n}{v_m \cdot t_{go}} \cdot c_d \cdot x_d \\
\dot{x}_d &= A_d \cdot x_d + b_d \cdot N \cdot v_m \cdot x
\end{aligned} \tag{3.12}$$

Where subscript n stands for nominal, subscript d stands for dynamics, $x = \omega$ is the LOS rate state variable, $x_d \in R^{n\times1}$ is the state vector of the missile dynamics with state matrix $A_d \in R^{n\times n}$ and vectors $b_d \in R^{n\times1}$, $c_d \in R^{1\times n}$, t_{go} is the remaining time to collision (time-to-go), v_m missile constant velocity magnitude, δ_n initial ideal missile lead angle, N constant navigation gain.

With initial conditions $x_0 \in R^{(n+1)\times 1}$ and the following expressions:

$$\begin{aligned} &\delta_n = sin^{-1}(sin(\theta_n) \cdot v_T/v_m) = \delta_0 - \delta_{err} \\ &t_{go} = t_f - t \\ &t_f = \rho_0/v_c \\ &v_c = v_m \cdot cos\,\delta_n - v_T \cdot cos\,\theta_n \end{aligned} \tag{3.13}$$

Where subscript n stands for nominal, subscript 0 denotes initial time t_0, v_m missile velocity, δ missile lead angle, δ_{err} initial heading error, v_T target constant velocity magnitude, v_c closing velocity, t_{go} is time-to-go, t_f is the flight time, t is the time that elapsed since the initial time t_0, ρ line of sight range.

Let x and x_d be unified with a single state vector x_{d_e}:

$$x_{d_e} = \begin{bmatrix} x \\ x_d \end{bmatrix} \in R^{n+1\times 1} \tag{3.14}$$

Where $x_{d_e} \in R^{n+1\times 1}$ is unified state vector, $x = \omega$ is the *LOS* rate state variable, $x_d \in R^{n\times 1}$ is the state vector of the missile dynamics.

3.3.2. *Linear Transformation*

The following notation is valid for the equations in the section:

A_d	$n \times n$ state matrix of the missile dynamics
a_c	Missile lateral acceleration input command
a_m	Missile lateral output acceleration
b_d	$n \times 1$ input vector of the missile dynamics
c_d	$1 \times n$ output vector of the missile dynamics
c_v	Constant $(cos\,\delta_n\,/v_m)$
I	Unit matrix of order n
N	Navigation gain
N'	Adjusted navigation gain
φ	Integration variable
s	Laplace variable
t	Time variable
T_e	$n + 1 \times n + 1$ linear transformation matrix
v_m	Missile velocity
x	*LOS* rate state variable (ω)
x_0	Initial condition of the *LOS* rate
x_d	$n \times 1$ state vector of missile dynamics
x_{d_0}	$n \times 1$ initial conditions vector of x_d
x_e	$n + 1 \times 1$ extended state vector
z	$n \times 1$ new state vector
z_0	$n \times 1$ initial conditions vector of z
z_e	$n + 1 \times 1$ extended equivalent state vector
z_{e_0}	$n + 1 \times 1$ initial conditions vector of z_e

System (3.12) can attain the mathematical form of (3.7) by the following linear transformation.

Based on the properties of the state transition matrix, vector x_d of the missile dynamics in (3.12) can be represented by the integral form:

$$x_d(t) = e^{A_d \cdot t} \cdot x_{d_0} + N \cdot v_m \cdot \int_0^t e^{A_d \cdot (t-\varphi)} \cdot b_d \cdot x(\varphi) \cdot d\varphi \tag{3.15}$$

Let $u = x(\varphi)$ and $dv = N \cdot v_m \cdot e^{A_d \cdot (t-\varphi)} \cdot b_d \cdot d\varphi$, integration by parts gives:

$$\begin{aligned} x_d(t) = e^{A_d \cdot t} \cdot x_{d_0} - N \cdot v_m \cdot x(t) \cdot A_d^{-1} \cdot b_d + N \cdot v_m \cdot x_0 \cdot e^{A_d t} \cdot A_d^{-1} \\ \cdot b_d + N \cdot v_m \cdot \int_0^t e^{A_d \cdot (t-\varphi)} \cdot A_d^{-1} \cdot b_d \cdot \dot{x}(\varphi) \cdot d\varphi \end{aligned} \tag{3.16}$$

Define the new variable z:

$$\begin{aligned} z(t) = e^{A_d t} \cdot \left(x_{d_0} + N \cdot v_m \cdot x_0 \cdot A_d^{-1} \cdot b_d\right) \\ + N \cdot v_m \cdot \int_0^t e^{A_d \cdot (t-\varphi)} \cdot A_d^{-1} \cdot b_d \cdot \dot{x}(\varphi) \cdot d\varphi \end{aligned} \tag{3.17}$$

In terms of $z(t)$, $x_d(t)$ is given by:

$$x_d(t) = z(t) - N \cdot v_m \cdot x(t) \cdot A_d^{-1} \cdot b_d \tag{3.18}$$

$z(t)$ in (3.17) is given in the general form of system-state response in terms of the matrix exponential $e^{A_d t}$. From this general form, the set of state equations for $z(t)$ is extracted as:

$$\begin{aligned} \dot{z}(t) &= A_d \cdot z(t) + N \cdot v_m \cdot \dot{x}(t) \cdot A_d^{-1} \cdot b_d \\ z_0 &= x_{d_0} + N \cdot v_m \cdot x_0 \cdot A_d^{-1} \cdot b_d \end{aligned} \tag{3.19}$$

Substitution of (3.18) and (3.19) back to (3.12) yields:

$$\begin{aligned} \dot{x}(t) &= \frac{2}{t_{go}} \cdot x(t) - \frac{\cos \delta_n}{v_m \cdot t_{go}} \cdot c_d \cdot (z(t) - N \cdot v_m \cdot x(t) \cdot A_d^{-1} \cdot b_d) \\ \dot{z}(t) &= A_d \cdot z(t) + N \cdot v_m \cdot \dot{x}(t) \cdot A_d^{-1} \cdot b_d \end{aligned} \tag{3.20}$$

Equivalently:

$$\begin{aligned} \dot{x}(t) &= \frac{2}{t_{go}} \cdot x(t) + \frac{\cos \delta_n}{t_{go}} \cdot N \cdot x(t) \cdot c_d \cdot A_d^{-1} \cdot b_d - \frac{\cos \delta_n}{v_m \cdot t_{go}} \cdot c_d \cdot z(t) \\ \dot{z}(t) &= N \cdot v_m \cdot \dot{x}(t) \cdot A_d^{-1} \cdot b_d + A_d \cdot z(t) \end{aligned} \tag{3.21}$$

With initial conditions $x_0 \in R^{1\times 1}$ for x and $z_0 = x_{d_0} + N \cdot v_m \cdot x_0 \cdot A_d^{-1} \cdot b_d \in R^{n\times 1}$ for z. Recall (1.17), the overall gain of the guidance loop has to preserve the navigation gain:

$$-c_d \cdot {A_d}^{-1} \cdot b_d = 1 \tag{1.17}$$

Define the adjusted navigation gain:

$$N' = cos\,\delta_n \cdot N \tag{3.22}$$

And let another constant:

$$cos\,\delta_n\,/v_m = c_v \tag{3.23}$$

c_v is scalar. Its inverse representation as $v_m/\,cos\,\delta_n = c_v^{-1}$ is just for presentation purposes.

(3.21) takes the form:

$$\begin{aligned} \dot{x}(t) &= \frac{2-N'}{t_{go}} \cdot x(t) - \frac{c_v}{t_{go}} \cdot c_d \cdot z(t) \\ \dot{z}(t) &= c_v^{-1} \cdot N' \cdot \dot{x}(t) \cdot A_d^{-1} \cdot b_d + A_d \cdot z(t) \end{aligned} \tag{3.24}$$

With x_0 and $z_0 = x_{d_0} + c_v^{-1} \cdot N' \cdot x_0 \cdot A_d^{-1} \cdot b_d$

With the substitution of $\dot{x}(t)$ and some arrangement, the equivalent form of (3.12) is given by:

$$\begin{aligned} \dot{x}(t) &= \frac{2-N'}{t_{go}} \cdot x(t) - \frac{c_v}{t_{go}} \cdot c_d \cdot z(t) \\ \dot{z}(t) &= A_d \cdot z(t) + \frac{N'}{t_{go}} \cdot z(t) + c_v^{-1} \cdot \frac{N'}{t_{go}} \cdot (2-N') \cdot A_d^{-1} \cdot b_d \cdot x(t) \end{aligned} \tag{3.25}$$

And initial conditions x_0 and $x_{d_0} + c_v^{-1} \cdot N' \cdot x_0 \cdot A_d^{-1} \cdot b_d$.

x and z can be unified with a single state vector z_e:

$$z_e = \begin{bmatrix} x \\ z \end{bmatrix} \in R^{n+1\times 1} \tag{3.26}$$

The relation of x_d and z as it appears in (3.18) can be expressed through the transformation matrix T_e:

$$z_e = T_e \cdot x_e \tag{3.27}$$

Where:

$$T_e = \begin{pmatrix} 1 & 0 \cdots 0 \\ N \cdot v_m \cdot A_d^{-1} \cdot b_d & 1 \end{pmatrix} \in R^{n+1\times n+1} \tag{3.28}$$

Since the determinant of T_e is nonzero ($det\,T_e = 1$), matrix T_e is nonsingular, having only time-constant parameters, systems (3.12) and (3.25) are internally equivalent with the same dynamic characteristics.

The next theorem analyzes the stability of system (3.25) whose state vector is the unified vector z_e.

3.4. System Analysis

Theorem 1

Let system (3.12) be represented in form (3.25), if

c.1. The dynamic matrix A_d is *Hurwitz*, and
c.2. $N' > 2$

Then the system is asymptotically stable in the sense of Definition 2.

Remark: Condition c.1 derives straightforwardly from the requirement of the system to be transformed into (3.25). Condition c.2 derives from inequality (3.41) and is well known for stability of proportional navigation systems [8] [7] [11].

Proof

The proof for stability of the equivalent system (3.25) is based on the Lyapunov candidate function:

$$v(z_e) = x^2(t) + z^T(t) \cdot H \cdot z(t) \tag{3.29}$$

Where $v(z_e)$ is a scalar function of vector argument z_e, $z_e \in R^{(n+1)\times 1}$ is the state vector of the equivalent system (3.25) with initials vector z_{e_0}, $x \in R^{1\times 1}$ is the *LOS* rate state, $z \in R^{n\times 1}$ is a state vector in the equivalent system (3.25) (superscript T stands for the transformation operator), and H is the solution of the Lyapunov equation:

$$A_d^T \cdot H + H \cdot A_d = -4 \cdot I \tag{3.30}$$

Where $A_d \in R^{n\times n}$ is state matrix of the missile dynamics and is the time invariant part of $z(t)$, I is unit matrix of order n.

The notations used for (3.29), (3.30), with the following, are valid for the rest of the section:

α_j	$j = 1, 2$ extremum functions of Lyapunov inequality terms
b_d	$n \times 1$ input vector of the missile dynamics
c_d	$1 \times n$ output vector of the missile dynamics

c_v	Constant ($cos\,\delta_n\,/v_m$)
λ_j	$j = 1..n$ eigenvalues of matrix H
N'	Adjusted navigation gain
t_f	Total flight time
t_{go}	Time-to-go, remained time to collision
τ_1	Positive – time index – constant
τ_2	Positive – time index – constant
v_m	Missile velocity
v_j	$j = 1,2,3$ temporal auxiliary functions

Let's take the time derivative of Lyapunov function v:

$$\dot{v} = 2 \cdot x \cdot \dot{x} + 2 \cdot z^T \cdot H \cdot \dot{z} \tag{3.31}$$

Substitution of $\dot{x}$ and $\dot{z}$ of the equivalent system (3.25) and arranging the terms in ascending order with respect to z yield:

$$\begin{aligned}\dot{v} = 2 \cdot \frac{2-N'}{t_{go}} \cdot x^2 - 2 \cdot \frac{c_v}{t_{go}} \cdot x \cdot c_d \cdot z \\ + 2 \cdot c_v^{-1} \cdot \frac{N'}{t_{go}} \cdot (2-N') \cdot x \cdot z^T \cdot H \cdot A_d^{-1} \cdot b_d \\ + 2 \cdot \frac{N'}{t_{go}} \cdot z^T \cdot H \cdot z + 2 \cdot z^T \cdot H \cdot A_d \cdot z\end{aligned} \tag{3.32}$$

The linear terms in z and those which are factors of $1/t_{go}$ are aimed now to be replaced with equivalent quadratic expressions as implied by v_1 to v_3:

$$\begin{aligned}v_1 = z^T \cdot z + 2 \cdot z^T \cdot \frac{c_v}{t_{go}} \cdot x \cdot c_d^T + \left(\frac{c_v}{t_{go}} \cdot x \cdot c_d^T\right)^T \cdot \left(\frac{c_v}{t_{go}} \cdot x \cdot c_d^T\right) \\ = \left(z + \frac{c_v}{t_{go}} \cdot x \cdot c_d^T\right)^T \cdot \left(z + \frac{c_v}{t_{go}} \cdot x \cdot c_d^T\right) \geq 0\end{aligned} \tag{3.33}$$

$$\begin{aligned}v_2 = z^T \cdot z - \ 2 \cdot c_v^{-1} \cdot \frac{N'}{t_{go}} \cdot (2-N') \cdot x \cdot z^T \cdot H \cdot A_d^{-1} \cdot b_d \\ + \left(c_v^{-1} \cdot \frac{N'}{t_{go}} \cdot (2-N') \cdot x \cdot H \cdot A_d^{-1} \cdot b_d\right)^T \\ \cdot \left(c_v^{-1} \cdot \frac{N'}{t_{go}} \cdot (2-N') \cdot x \cdot H \cdot A_d^{-1} \cdot b_d\right)\end{aligned} \tag{3.34}$$

$$= \left(z - c_v^{-1} \cdot \frac{N'}{t_{go}} \cdot (2 - N') \cdot x \cdot H \cdot A_d^{-1} \cdot b_d\right)^T \cdot \left(z - c_v^{-1} \cdot \frac{N'}{t_{go}} \cdot (2 - N') \cdot x \cdot H \cdot A_d^{-1} \cdot b_d\right) \geq 0$$

$$\begin{aligned} v_3 &= z^T \cdot z - 2 \cdot \frac{N'}{t_{go}} \cdot z^T \cdot H \cdot z + \left(\frac{N'}{t_{go}} \cdot H \cdot z\right)^T \cdot \left(\frac{N'}{t_{go}} \cdot H \cdot z\right) \\ &= \left(z - \frac{N'}{t_{go}} \cdot H \cdot z\right)^T \cdot \left(z - \frac{N'}{t_{go}} \cdot H \cdot z\right) \geq 0 \end{aligned} \tag{3.35}$$

Substitution of $v_{1,2,3}$ with the subtraction of the extra terms brings us to the following form of the Lyapunov derivative:

$$\begin{aligned} \dot{v} = {} & 2 \cdot \frac{2 - N'}{t_{go}} \cdot x^2 - v_1 + z^T \cdot z + \left(\frac{x \cdot c_v}{t_{go}}\right)^2 c_d \cdot c_d^T - v_2 + z^T \cdot z \\ & + \left(c_v^{-1} \cdot \frac{N'}{t_{go}} \cdot (2 - N') \cdot x\right)^2 \cdot (H \cdot A_d^{-1} \cdot b_d)^T \\ & \cdot (H \cdot A_d^{-1} \cdot b_d) - v_3 + z^T \cdot z + \left(\frac{N'}{t_{go}}\right)^2 \cdot (H \cdot z)^T \cdot (H \cdot z) \\ & + 2 \cdot z^T \cdot H \cdot A_d \cdot z \end{aligned} \tag{3.36}$$

Now, the last term in (3.36) equals the left-hand side of the Lyapunov equation (3.30) that can be introduced here to yield:

$$\begin{aligned} \dot{v} = {} & -v_1 - v_2 - v_3 - z^T \cdot z + 2 \cdot \frac{2 - N'}{t_{go}} \cdot x^2 + \left(\frac{x \cdot c_v}{t_{go}}\right)^2 c_d \cdot c_d^T \\ & + \left(c_v^{-1} \cdot \frac{N'}{t_{go}} \cdot (2 - N') \cdot x\right)^2 \cdot (H \cdot A_d^{-1} \cdot b_d)^T \\ & \cdot (H \cdot A_d^{-1} \cdot b_d) + \left(\frac{N'}{t_{go}}\right)^2 \cdot (z^T \cdot H^T \cdot H \cdot z) \end{aligned} \tag{3.37}$$

From this last step let's withdraw the first three non-negative terms to arrive at the following inequality:

$$\begin{aligned}&\dot{v}\\&\leq\ -z^T\cdot z+2\cdot\frac{2-N'}{t_{go}}\cdot x^2+\left(\frac{x\cdot c_v}{t_{go}}\right)^2 c_d\cdot c_d^T\\&+\left(c_v^{-1}\cdot\frac{N'}{t_{go}}\cdot(2-N')\cdot x\right)^2\cdot(H\cdot A_d^{-1}\cdot b_d)^T\cdot(H\cdot A_d^{-1}\cdot b_d)+\left(\frac{N'}{t_{go}}\right)^2\\&\cdot(z^T\cdot H^T\cdot H\cdot z)\end{aligned}\tag{3.38}$$

Which holds for any time on the interval (for further details on the relation of t_{go} and τ_1 see section 3.2):

$$0\leq\tau_1\leq\ t_{go}\tag{3.39}$$

The following steps are intended for the simplification of the inequality form of (3.38).

The first simplification uses these positions:

1. $0\leq\lambda_1\leq\lambda_2\ ...\leq\lambda_n$ are the eigenvalues of matrix H, based on the properties of H as symmetric and positive, they exist:

$$0\leq\lambda_1\cdot z^T\cdot z\leq z^T\cdot H\cdot z\leq\lambda_n\cdot z^T\cdot z\tag{3.40}$$

2. Condition c.2 ($N'>2$ or $N'-2>0$).
3. $\tau_1\leq t_{go}$

Applying these three, the first two terms in (3.38) reduce to the form:

$$\begin{aligned}-z^T\cdot z-2\cdot\frac{N'-2}{t_{go}}\cdot x^2&\leq-\frac{1}{\lambda_n}\cdot z^T\cdot H\cdot z\ -2\cdot\frac{N'-2}{t_{go}}\cdot x^2\\&=\frac{-\frac{t_{go}}{\lambda_n}\cdot z^T\cdot H\cdot z-2\cdot(N'-2)\cdot x^2}{t_{go}}\\&\leq\frac{-\frac{\tau_1}{\lambda_n}\cdot z^T\cdot H\cdot z-2\cdot(N'-2)\cdot x^2}{t_{go}}\\&\leq\ -min\left(\frac{\tau_1}{\lambda_n},2\cdot(N'-2)\right)\cdot\frac{v(x_e)}{t_{go}}\end{aligned}\tag{3.41}$$

Let's denote this last term with:

$$\alpha_1=min\left(\frac{\tau_1}{\lambda_n},2\cdot(N'-2)\right)\tag{3.42}$$

Second simplification, pertaining the last three terms in (3.38), is achieved by determining their supremum with:

$$\alpha_2 = max\left(\frac{N'^2}{\lambda_1}, c_v^2 \cdot c_d \cdot c_d^T + \left(c_v^{-1} \cdot N' \cdot (2 - N')\right)^2 \cdot (H \cdot A_d^{-1} \cdot b_d)^T \cdot (H \cdot A_d^{-1} \cdot b_d)\right) \tag{3.43}$$

When α_2 is multiplied by $v(z_e)/t_{go}^2$.

Notice that since $N' > 2$, according to condition c.2 of the theorem, the constants α_1 and α_2 have positive values.

The first argument of (3.43) is a reduction of the last term of (3.38):

$$\left(\frac{N'}{t_{go}}\right)^2 \cdot (z^T \cdot H^T \cdot H \cdot z) = N'^2 \cdot \frac{z^T \cdot H^2 \cdot z}{t_{go}^2} \leq \frac{N'^2}{\lambda_1} \cdot \frac{z^T \cdot H \cdot z}{t_{go}^2} \tag{3.44}$$

All the above yields the final form of the Lyapunov inequality:

$$\dot{v}(z_e) \leq \left(\frac{\alpha_2}{t_{go}^2} - \frac{\alpha_1}{t_{go}}\right) \cdot v(z_e) \tag{3.45}$$

To solve (3.45), divide both sides by $v(z_e)$ and integrate with respect to t:

$$v(z_e) \leq e^{\alpha_2 \cdot \left(\frac{1}{t_{go}} - \frac{1}{t_f}\right)} \cdot \left(\frac{t_{go}}{t_f}\right)^{\alpha_1} \cdot v(z_{e_0}) \tag{3.46}$$

On $\tau_1 \leq t_{go}$, $v(z_e)$ attains a boundary by strengthening the main inequality solution (3.46) with:

$$\max_{\tau_1 \leq t_{go}} v(z_e) \leq e^{\frac{\alpha_2}{\tau_1}} \cdot v(z_{e_0}) \tag{3.47}$$

Which approves the system is uniformly stable in the sense of Definition 1.

On $\tau_1 \leq t_{go} \leq \tau_2$, $v(z_e)$ attains asymptotic convergence, with respect to t_f, by strengthening (3.46) with:

$$\max_{\tau_1 \leq t_{go} \leq \tau_2} v(z_e) \leq e^{\frac{\alpha_2}{\tau_1}} \cdot \left(\frac{\tau_2}{t_f}\right)^{\alpha_1} \cdot v(z_{e_0}) \tag{3.48}$$

Which approves the system is asymptotically stable in the sense of Definition 2.

Hence, Theorem 1 is proven.

Derivations from Theorem 1 – Numerical Analysis

During the proof of Theorem 1 the constants α_1 and α_2 were derived. The conditions prove their values to be positive. Finally these constants determine both the inequalities for uniform stability and for asymptotic stability. Therefore it is important, from a system design point of view, to analyze the properties of the stability results in terms of these constants and their dependence on the system parameters.

Let: $\tau_1 = 0.1, \tau_2 = 1, N' = 3$. With these substitutions, (3.42) becomes:

$$\alpha_1 = \frac{0.1}{\lambda_n} \tag{3.49}$$

Where α_1 positive constant and λ_n highest eigenvalue of matrix H – the solution of the Lyapunov equation (see 3.30).

As inequality (3.48) states, the asymptotic convergence with respect to t_f grows with powers of α_1. It is simple therefore to see that the rate of convergence is a function of the system parameters that solve the Lyapunov equation (3.30).

For a first example let A_d be a state-matrix of a first order system:

$$A_d = -\frac{1}{\tau_m} \tag{3.50}$$

Where A_d state-matrix of a first order system and τ_m is a positive time constant.

As $\tau_m > 0$ the system is Hurwitz. Solving (3.30) gives $H = 2 \cdot \tau_m$, which is also the maximum and the only eigenvalue; the change of α_1 with change of τ_m is shown in Figure 3.2:

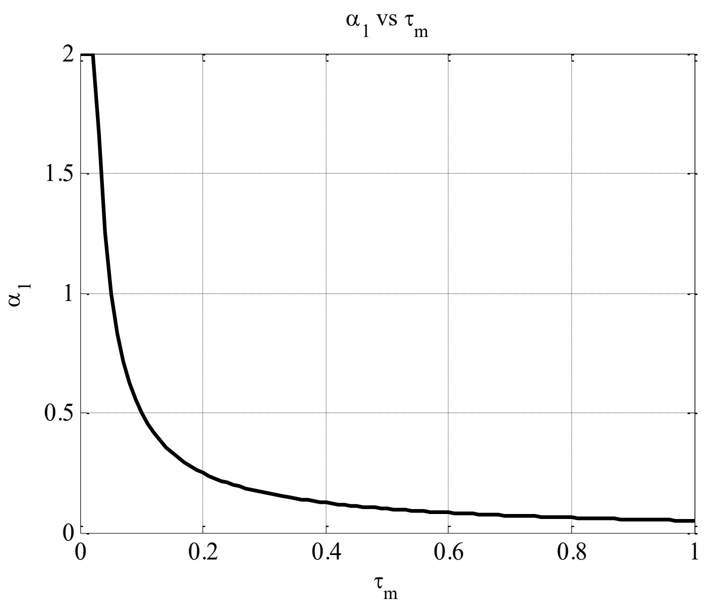

Figure 3.2: α_1 vs τ_m for 1st order system

At the points $\lambda_n = 2 \cdot \tau_m = 0.1, 0.5, 1$, the values of α_1 are $0.5, 0.1, 0.05$ respectively. For these values of α_1, the expression $\left(\frac{1}{t_f}\right)^{\alpha_1}$ which determines the rate of convergence for the system appears in Figure 3.3:

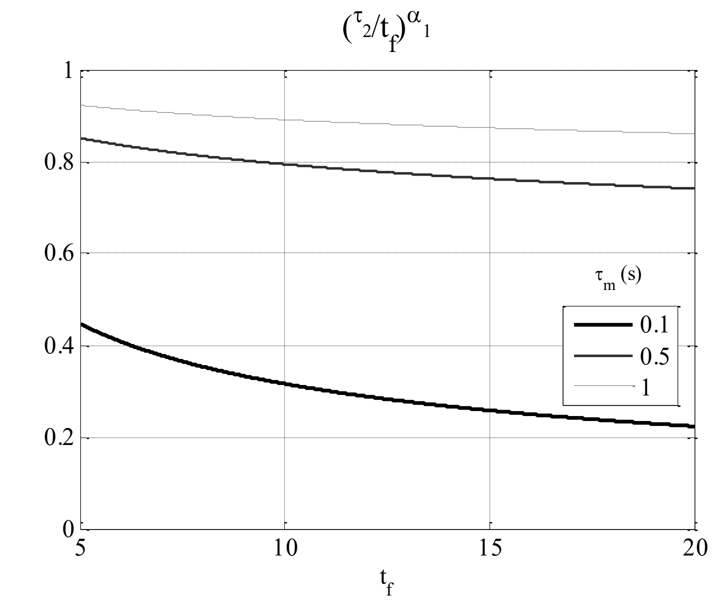

Figure 3.3: Convergence rate vs t_f vs α_1 for 1st order system

According to Figure 3.3 the convergence is obviously faster for higher values of α_1, which happen for shorter time constants τ_m.

In the second example a missile with dynamics of second order is engaged. Now A_d is given by:

$$A_d = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2 \cdot \zeta \cdot \omega_n \end{pmatrix} \tag{3.51}$$

Where A_d state matrix of second order system, ω_n natural frequency and ζ damping ratio.

Here again positive ω_n and ζ prove left-half poles, whereas the solution of the Lyapunov equation (3.30) is now more complicated:

$$H = \begin{pmatrix} \dfrac{1 + 4 \cdot \zeta^2 + \omega_n^2}{\zeta \cdot \omega_n} & \dfrac{2}{\omega_n^2} \\ \dfrac{2}{\omega_n^2} & \dfrac{1 + \omega_n^2}{\zeta \cdot \omega_n^3} \end{pmatrix} \tag{3.52}$$

Where H is the solution of Lyapunov equation (3.30), ω_n natural frequency and ζ damping ratio.

To examine α_1 with change of ζ we set $\omega_n = 1\ r/s$. The results are illustrated in Figure 3.4:

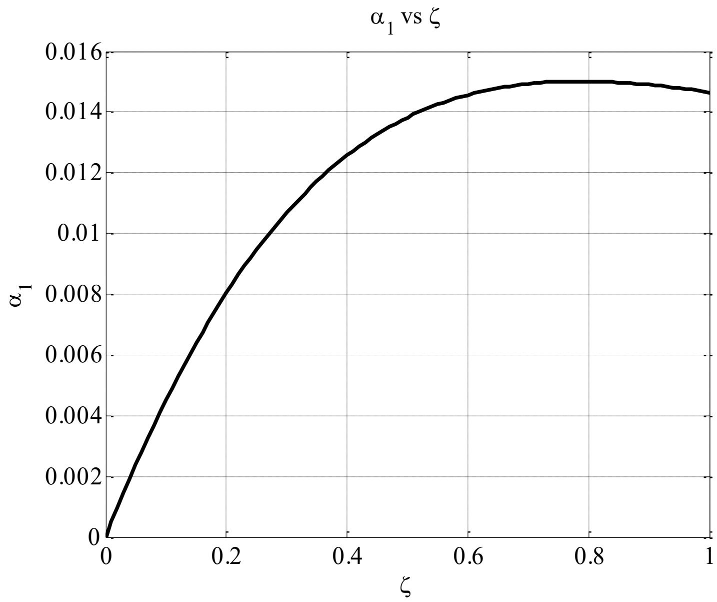

Figure 3.4: α_1 vs ζ for 2nd order system

As presented earlier, the higher α_1 provides better results in terms of quick convergence. Then a natural candidate-point for best performance is $= 0.8$, where the line in Figure 3.4 reaches its maximum.

The performances of the system with respect to $\zeta = 0.8$, and for comparison purposes also $\zeta = 0.2$ and $\zeta = 1$, appear in Figure 3.5. Since the rate of convergence of the second order system is quite lower with respect to the first order, then the range of values in Figure 3.5 is focused in the region where variations of the function take place.

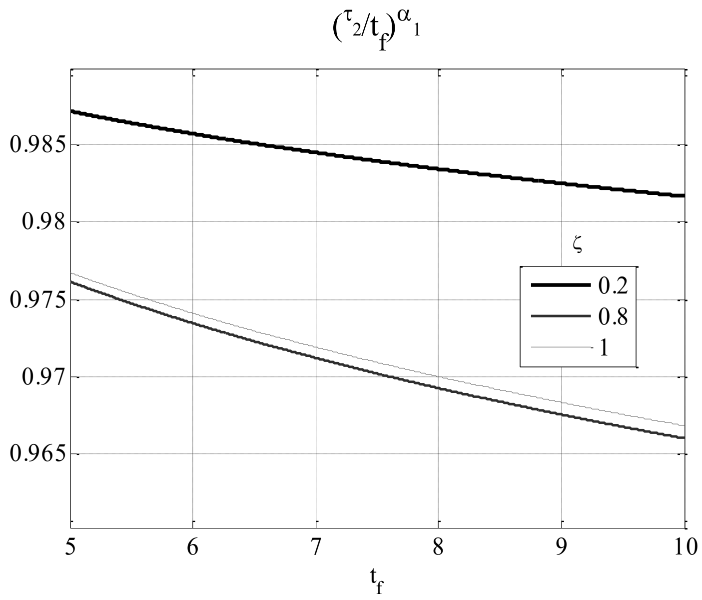

Figure 3.5: Convergence vs t_f vs α_1 for 2nd order system

Indeed, $\zeta = 0.8$ provides best results in terms of asymptotic convergence.

The analysis of the stability as exemplified here, and was derived from the results of the extended Lyapunov approach, is a powerful tool for the design of guidance systems, as it provides a straightforward indication not only to the system stability but also for best parameters selection in terms of rate of convergence.

3.5. Summary

Chapter 3 introduced the analysis of the guidance system with an extended approach of Lyapunov stability. This approach was found significant because it provides tools to investigate time-varying systems which present a singular point. Based on that method we investigated the principle equation of the linearized system, namely the relation of the acceleration command to the line of sight rate derivative. Now that this problematic subsystem proved to have conditions for asymptotic stability, we can move to the next chapter to consider the other relations of the acceleration command and the system states.

4. Miss Distance Analysis

The differential equation that connects the missile acceleration command with the line of sight rate was investigated in the previous chapter and was found to have conditions for attaining asymptotic stability. Still there remains to consider the other states, which can be viewed as system output, since their final values determine the miss distance. To complete this investigation, the following chapter introduces stability-analysis of the subsystem that possesses the missile-target relative position.

4.1. Miss Distance Evaluation

The ultimate objective of a guided missile is to disable or destroy the target. But inaccuracies, limitations and time-lags affect the missile as to flying close to the target but not necessarily hitting it directly. With the help of high explosives, near misses can be converted to successful intercepts. For that reason the guidance system has to cause the miss-distance to be as small as possible. To produce reliable evaluation of the performance of the guidance system the miss-distance measure and the exact time that it occurs, have to be considered separately.

4.1.1. Computational Procedure

When running an ideal system the moment in which the simulation should stop is easily evaluated. As we saw in section 2.1 when ideal conditions exist the total flight time is linear with respect to the distance to the target:

$$t_f = \rho_0 / v_c \tag{2.5}$$

Where t_f is the flight time from the measure of the range until collision, ρ_0 line of sight range at fixed moment t_0, v_c closing velocity.

But for the general case the conditions to stop the guidance must get real-time inputs of the distance to the target to announce miss-distance time. In addition, the way to measure the miss-distance is at the most a question of the interception objective. However, the view common for everyone is that the missile guidance has to close, as much as possible, the distance separating it with the target. Then the criterion at the most general level, for stopping the guidance simulation, is based on the definition that states that the miss-distance is the closest approach of the missile to the target. An equivalent definition can be found in the Military Handbook of Missile Flight Simulation[1] and the drawing in Figure 4.1 was borrowed from there.

Another important challenge is the accuracy of the miss-distance calculation in the numerical simulation. As already shown extensively, the system introduces a singular point in the vicinity of the impact point. Numerically it is a major source of errors and the treatment of it depends on the engaged model and the simulation platform.

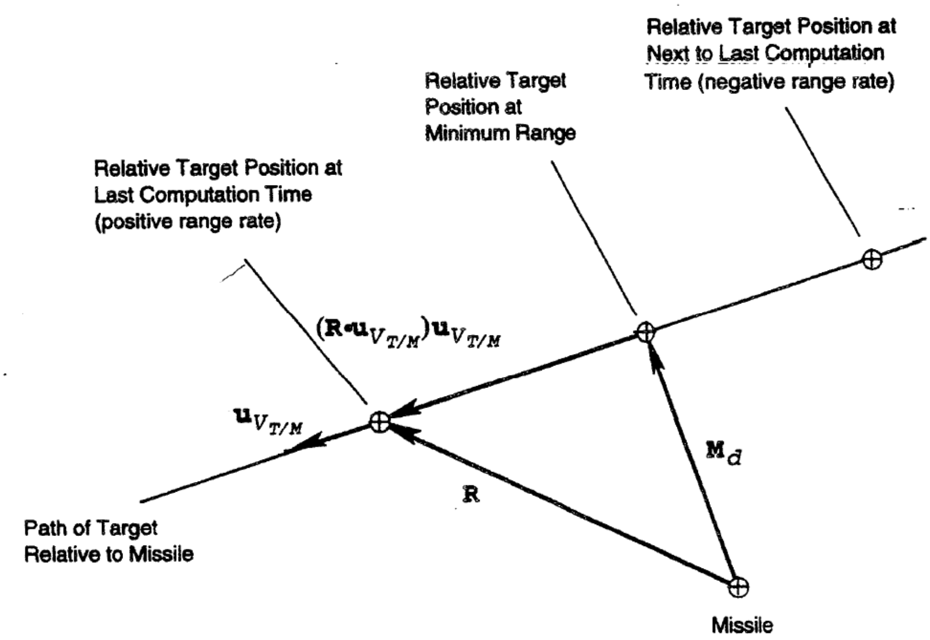

Figure 4.1: Miss-distance vector diagram (taken from Military Handbook of Missile Flight Simulation). The descriptions in the handbook for the symbols in the illustration are: M_d miss distance vector at time of closet approach directed from missile to target, R range vector, $u_{V_{T/M}}$ unit vector in direction of relative velocity vector $V_{T/M}$.

In the view of computational procedure, the miss-distance in the sense that is given above, occurs when the line of sight vector r (see equation (1.2)) reaches a minimum. Applying it includes cyclic calculation of the range rate $\dot{\rho}$ and catching the time of sign-changing, namely the time when the range changes from decreasing to increasing. This procedure serves us to analyze the miss-distance in the following sections.

[1] Missile Flight Simulation Part One Surface-to-Air Missiles, MIL-HDBK-1211(MI) 17 July 1995. In: Military Handbook. 1995

4.1.2. Miss Distance Output

Let's now return to section 3.3 where some arrangement provided us with the following form of the linearized system:

$$\begin{aligned}
\dot{x}_1 &= v_T \cdot \sin\theta_n \cdot x_5 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= \frac{2}{t_{go}} \cdot x_3 - \frac{c_m}{t_{go}} \cdot c_d \cdot x_d \\
\dot{x}_5 &= \frac{1}{v_m} \cdot c_d \cdot x_d \\
\dot{x}_d &= A_d \cdot x_d + b_d \cdot a_c
\end{aligned} \tag{4.1}$$

Where subscript n stands for nominal, subscript d stands for dynamics, $[x_1, x_2, x_3, x_5] = [\rho, \lambda, \omega, \gamma_m]$ are the state variables, $x_d \in R^{n\times 1}$ is the state vector of the missile dynamics with state matrix $A_d \in R^{n\times n}$ and vectors $b_d \in R^{n\times 1}$, $c_d \in R^{1\times n}$, a_c input of the missile acceleration, $c_m = \cos\delta_n / v_c$ is a constant coefficient, t_{go} is the remaining time to collision (time-to-go), v_c is the constant closing velocity, v_m missile constant velocity magnitude, θ_n initial target aspect angle and δ_n initial ideal missile lead angle.

Initial conditions to (4.1) are given by:

$$\begin{aligned}
x_{1_0} &= 0 \\
x_{2_0} &= 0 \\
x_{3_0} &= (v_T \cdot \sin\theta_0 - v_m \cdot \sin\delta_0)/(v_c \cdot t_f) \\
x_{4_0} &= 0 \\
x_{5_0} &= \delta_{err} \\
x_{d_0} &= 0
\end{aligned} \tag{4.2}$$

Where $x_0 \in R^{(n+5)\times 1}$ is the initial conditions vector of system to (4.1) , v_T and v_m target and missile velocities, θ_0 is the target aspect angle at time t_0 and δ_0 is the missile lead angle at time t_0, δ_{err} initial heading error – the deviation of the missile lead angle from the correct direction (see 1.9), x_d state vector of missile dynamics of n^{th} order, v_c is the constant closing velocity and t_f is the flight time.

(4.2) possesses the vector of initial conditions for (4.1) but in chapter 2 it was shown that for numerical analysis of the linear system, the reference solution about which the linearization was done, has to be summed with the solution of (4.1) (4.2).

Examining (4.1), we distinguished three subsystems that share some variables with each other: the closed-loop system – that includes all the factors of t_{go}, from the missile acceleration to the line of sight angle rate ω. A second subsystem from ω to λ. A third subsystem from a_m to ρ. The second and third subsystems contain a single integrator and two integrators in the output of the closed-loop system respectively. Then even though it was shown in the previous chapter that the closed-loop system with ω within, is asymptotically stable, that it is not yet guaranteed that the third system, containing the missile-target range ρ, is stable. Even the stability of the second system containing λ is uncertain. On the other hand the integration units are not sufficient conditions to deny its stability, since the system is time-varying, the laws of time-invariant systems are not applicable to it.

For each subsystem at the output of the closed-loop system we will try to determine stability in an input-output sense and we'll do that by numerical simulation, by determining the input and inspecting $y = \lambda$ and $y = \rho$ at the outputs. The theoretical basis for the analysis is given in the following definitions.

4.2. Input – Output Stability

Recall the nomenclature:

t	Time variable
t_f	Total flight time of a single process
t_{go}	$t_f - t$, Time-to-go, remaining time to collision
τ_1	Positive constant
τ_2	Positive constant ($\tau_2 > \tau_1$)

Definition 1

System (4.1) is input-output stable from input u to output y if its own motion remains bounded on the time interval $\tau_1 \leq t_{go} \leq \tau_2$ as $t_f \to \infty$.

Definition 2

System (4.1) is asymptotically stable from input u to output y if it is input-output stable and its own motion tends to zero on the time interval $\tau_1 \leq t_{go} \leq \tau_2$ as $t_f \to \infty$.

Notes on Definition 1 and Definition 2:

n.1. The notion of the time parameters τ_1, τ_2, was illustrated in Figure 3.1 (section 3.2). Recall also that as elucidated there, a single unit of time in the process is the time-length of a flight, therefore the sequence of final values $y = x(t_f)$ with respect to the flight-times t_f is the process we want to test.

n.2. Input in Definition 1 and Definition 2 refers to any input signal or set of initial conditions. It should be noted that any initial condition x_{m_0} can be represented as an input signal by:

$$u = I_m \cdot x_{m_0} \cdot \delta(t) \tag{4.3}$$

Where u is an input signal equivalent to the desired initial condition, I_m is the m-th column of the unit matrix, x_{m_0} is the desired initial condition of the m-th variable, $\delta(t)$ is the Dirac delta function, t time variable.

When the input is of initial conditions type, the sense of own motion of a system is its reaction to initial conditions, i.e. its free response.

n.3. In the following tests the miss-distance analysis is performed with respect to an input which is initial conditions.

4.2.1. Miss Distance Analysis

Following Definition 1 and Definition 2, let the system input be any initial conditions vector $u = x_0$ such that the initial state of $x_5 = \gamma_m$ is nonzero, $x_{5_0} = \delta_{err} \neq 0$.

Referring back to the early chapters, δ_{err} is the deviation of the missile velocity from collision course with the target. $x_{5_0} = \delta_{err} \neq 0$ also means $x_{3_0} = \omega_0 \neq 0$ but from a physical point of view the deviation from collision course is led by an error in the velocity direction.

Initial heading error is one of the main theoretical problems in missiles guidance. Target maneuvering and high-order dynamics are also central in this regard. The proposed analysis when examined with output $y = \rho(t_f)$ explores the miss-distance with respect to heading error. The dynamics problem also finds an answer here. In opposition, the target in the following investigations is assumed as non-maneuvering.

1st Heading Error

Let:

$$\delta_{err} = -15° \tag{4.4}$$

Where δ_{err} is the heading error.

And let the system output be the sequence of last line of sight angles $y = \lambda(t_f)$, for unbounded increase of t_f.

The importance of the system behavior with respect to this output, arises from its position one integrator after the closed-loop system, for which conditions to asymptotic stability were found in the previous chapter. Lack of conditions for stability of that subsystem makes useless the search for stability conditions of the system with output $y = \rho(t_f)$, which possesses two integrators after the closed-loop system.

Accordingly, Figure 4.2 presents the final *LOS* angle for increasing initial range, which is equivalent to an increment in t_f, for $\delta_{err} = -15°$, and for three orders of dynamics of the linearized model. To view the results with respect to zero, the theoretical value of the *LOS* angle at t_f was subtracted from the results of the simulation.

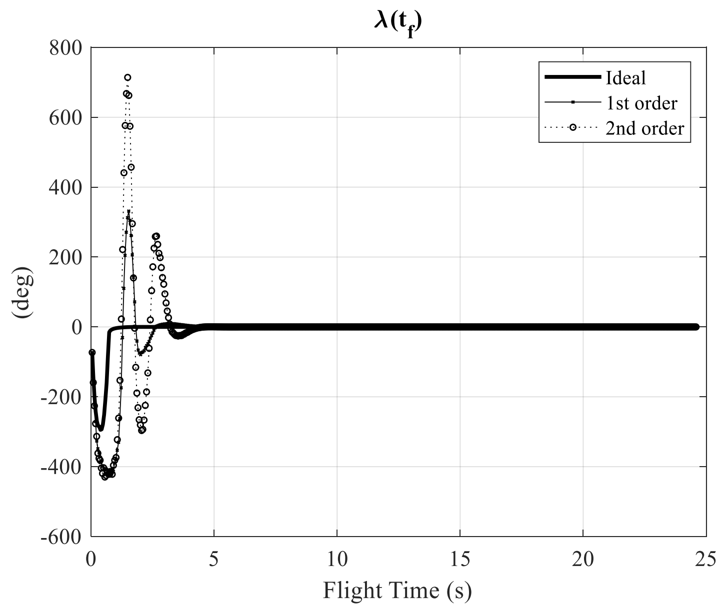

Figure 4.2: Final *LOS* angle with respect to t_f for heading error $\delta_{err} = -15°$, for ideal dynamics and dynamics of 1^{st} and 2^{nd} order, linear system.

Figure 4.2 shows that the response of the system in the transition-time is characterized by the dynamics part. It also shows that as the flight time increases the system converges into a small value around zero. But the important question is the behavior around zero, adherence to zero means the system is asymptotic stable and otherwise, not. With that regard, in Figure 4.3 is displayed a focused view of Figure 4.2, in an arbitrary interval of time, after the initial overshoot.

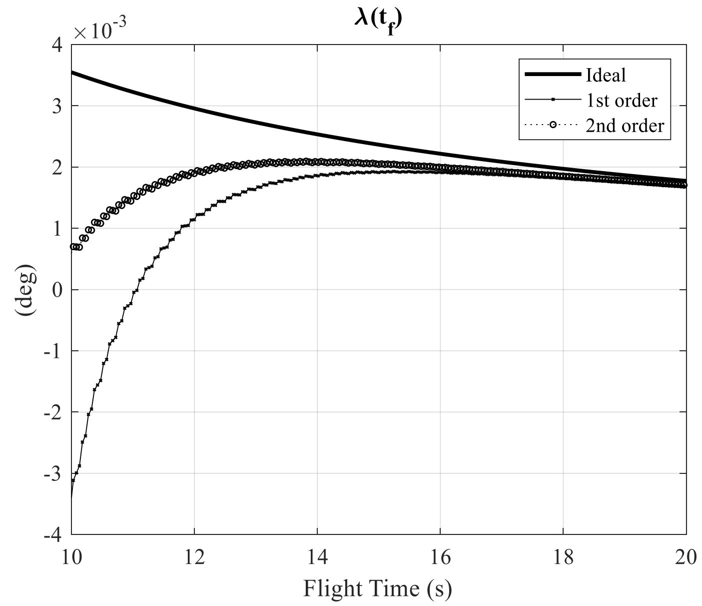

Figure 4.3: Focused view of the final *LOS* angle with respect to t_f for heading error $\delta_{err} = -15°$.

By Figure 4.3 it is straightforwardly concluded that system (4.1) is asymptotic stable with respect to input x_{5_0} and output $y = \lambda(t_f)$.

Let now the output be the miss-distance for increasing flight times, $y = \rho(t_f)$. In Figure 4.4 are displayed results for $\delta_{err} = -15°$, for three orders of dynamics of the linearized model.

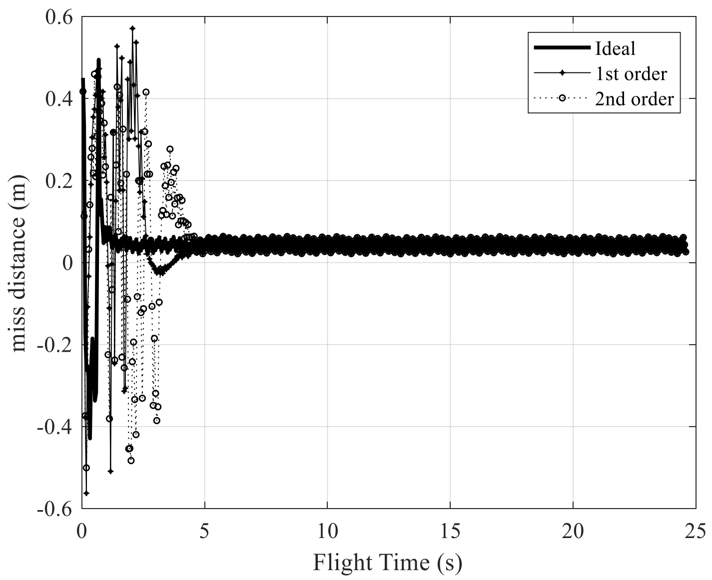

Figure 4.4: Miss-distance with respect to t_f for heading error $\delta_{err} = -15°$, for ideal dynamics and dynamics of 1st and 2nd order, linear system.

Figure 4.4 presents that the miss-distance, for each order of missile dynamics, converges to a small value close to zero, but not zero. A small constant gain, keeps the miss-distance from zero that two integrators cannot close.

Figure 4.4 shows that system (4.1) is stable with respect to input $u = x_o$ with $x_{5_0} = -15°$ and output $y = \rho(t_f)$, but not asymptotic stable with respect to the same conditions.

2nd Heading Error

Additional tests will be performed now for second heading error. Let:

$$\delta_{err} = -7.5° \tag{4.5}$$

Where δ_{err} is the heading error.

Figure 4.5 and Figure 4.6 introduce the performances of the system with respect to $y = \lambda(t_f)$ and $y = \rho(t_f)$ respectively. Figure 4.5 shows that similarly to the first heading error the system is asymptotic stable with respect to the *LOS* angle. Figure 4.6 shows that with respect to the miss distance the system is stable but not asymptotic stable.

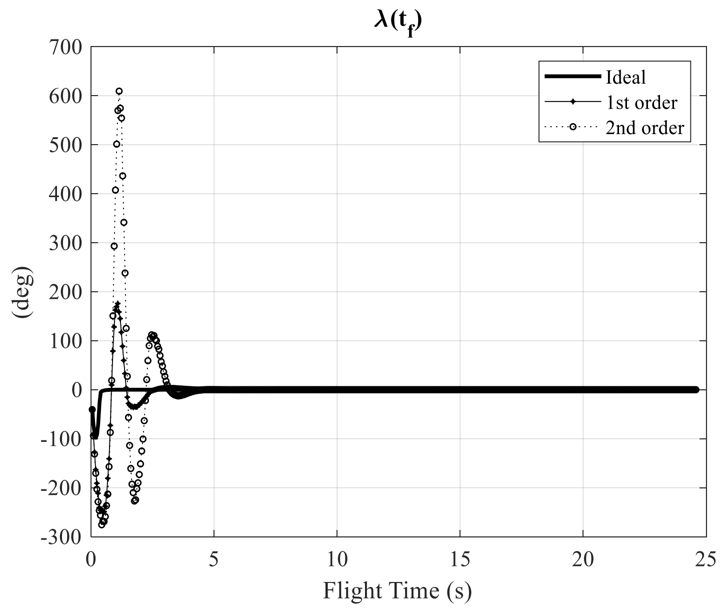

Figure 4.5: Final *LOS* angle with respect to t_f for heading error $\delta_{err} = -7.5°$, for ideal dynamics and dynamics of 1st and 2nd order, linear system.

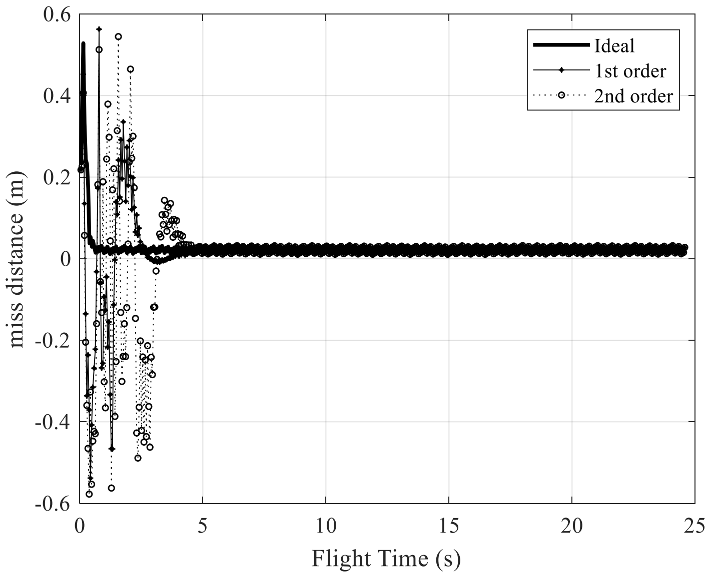

Figure 4.6: Miss distance with respect to t_f for heading error $\delta_{err} = -7.5°$, for ideal dynamics and dynamics of 1st and 2nd order, linear system.

Let a third heading error, $\delta_{err} = -30$. Figure 4.7 introduces the miss distance with respect to the three different heading errors for linear system with ideal dynamics.

The steady state is our interest, which indicates that as the absolute value of the input is smaller so the constant error in the output is smaller, i.e. smaller miss-distance. Then the system with respect to miss distance output is stable about input $\delta_{err} \neq 0$ and is bounded with respect to a bounded change of the same input.

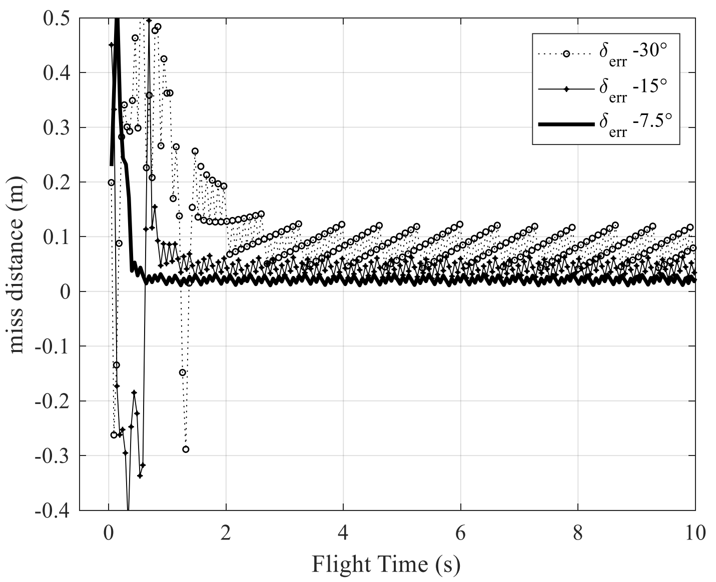

Figure 4.7: Miss distance comparison for three different heading errors.

This is to note, the improvement of the results, or at least the preservation of the results, with respect to increase in flight time, which is equivalent to an increase of the range to target, reflects the stability characteristic of the guidance system. In general the performances of a missile may deteriorate with the increase in target range, because effects that were not considered here, such as combustion time and friction forces that slow the velocity down. In this case the conditions between one run to another are no longer similar and the stable character of the system becomes obscured by other phenomena.

4.2.2. Summary

Chapter 4 introduced an analysis of the guidance system with respect to miss-distance. A program based on the linearized model as developed in chapter 2, simulated the system in a series of conditions, distinct from each other by an increase of target range. It was demonstrated that for a given heading error serves as input, and miss distance as output, the system is stable in an input-output sense. It was also demonstrated that with respect to the same conditions the system is not asymptotic stable.

5. Summary and Conclusions

n.1. The above work studies guidance systems employing a proportional navigation (*PN*) control law. The stability characteristics of the guidance system are at the center of the study. Starting with missile-target motion and ending with actual lateral acceleration of the missile, the guidance loop includes kinematics, guidance law and missile dynamics, each of them imposes its unique character on the overall system.

n.2. The reference trajectory used for linearization is ideal collision-triangle conditions where the missile and the target travel in constant velocities, in straight lines. The derived model is time-varying.

n.3. In the linearized model it is possible to distinguish an LTI part, which consists of the missile dynamics and the linearized kinematics, and time-varying part, consisting of a closed-loop about the LOS rate. The time-varying part depends on inverse terms of t_{go} which introduce a singular point at the vicinity of the impact point. Furthermore, two integrators separate the most interesting variable, i.e. the missile-target range, from the time-varying subsystem.

n.4. Conditions for asymptotic stability of the closed-loop subsystem, from the acceleration command to the line of sight (LOS) angle rate ω, are obtained. Due to the singular point that is present in ω's derivative, the stability of the system is investigated by an extended approach to Lyapunov, which examines stability of closed-loop systems with respect to unbounded increase of the total time t_f. To apply this approach, the system has been transformed linearly to an equivalent form, as required by the theory.

n.5. Numerical simulation examined the stability of the overall system. Stability here is in input-output sense and it reflects the behavior of the miss-distance. While the results prove initial convergence of the miss-distance with respect to an ascending initial target range, later on the miss-distance oscillates about some small constants, where it is hard to determine what part of it is a natural response of the system and what is contributed by numerical errors. This is the weakness of the numerical analysis. But this approach was employed here because of the irrelevance of other methods.

n.6. The results of the investigations so far indicate that asymptotic stability with respect to miss-distance in input-output sense, may actually be achieved with some compensation for the second integrator in the series after the closed-loop subsystem. The study of this feedback loop is not of the scope of the work but most of the material required for its development is given here.

n.7. While an asymptotic convergence of the miss-distance is not guaranteed, it was demonstrated to be bounded. Together with an analytic tool to adjust the rate of convergence of the acceleration command, the designer is provided with an advanced tool to develop the guidance system.

n.8. The results derived in the work are applicable to pure proportional navigation (PPN), and even though the relative proximity of the true proportional navigation (TPN) version, the equations of motion should differ when the acceleration command is exerted not only in the normal component of the velocity, and the case then has to be considered separately. Regarding that, there are more difficult versions of PN, as the different biased commands, as Shneydor surveys in chapter 7 of his book, which definitely cannot be concluded from this work by similarity, and they require a study of their own.

n.9. Finally, all the different subsystems of the missile as seeker and autopilot are considered here by linear time-invariant dynamics. In fact many of the challenges that a designer faces are in biases, saturations and other nonlinearities of the sensors and of the seeker above all. The treatment to this kind of errors, is in general within a complete solution which considers not just stability but also state-estimation and adaptive or optimal control elements in the guidance system.

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Appendix 1

The numerical analyses and the examples in this research are based on proportional navigation guidance simulation. The simulation program is written in Matlab and makes use of *ode45* for the calculation of derivatives in the nonlinear model and *ode23s* in the linear model.

The simulation suggests two options for missile dynamics – ideal and first order, both given in terms of state-space models. User input sets the initial conditions of the target initial range and the missile initial heading error with respect to the line of sight.

Selective parts of the nonlinear model simulation are submitted here to the reader.

Initialization

```
%
%   Time-Constant Parameters
%
Deg2Rad =    pi / 180;

%
%   Input Initialization
%
xT0      =    rng0;
yT0      =    0;
xM0      =    0;
yM0      =    0;
deltaErr=    deltaErrDeg * Deg2Rad;

%  Initial Values of State Vector and Algebraic Variables
%   X = [1-Rmt, 2-Lam, 3-Omg, 4-GamT, 5-GamM, 6-aM]
%  Algebraic Variables
%   Alg = [1-aMc,2-Del,3-Thet, 4-vC]
Rmt0     =    sqrt((xT0 - xM0)^2 + (yT0 - yM0)^2);       %X(1)
Lam0     =    asin((yT0 - yM0)/Rmt0);                     %X(2)
Thet0    =    TheTDeg * Deg2Rad;                          %Alg(3)
GamT0    =    Lam0 + Thet0;                               %X(4)

DelCorrect  =    asin(vT / vM * sin(GamT0 - Lam0));
Del0        =    DelCorrect + deltaErr;                   %Alg(2)

GamM0    =    Lam0 + Del0;                                %X(5)
Omg0     =    (vT * sin(Thet0) - vM * sin(Del0)) / Rmt0;%X(3)
aM       =    0;                                          %X(6)

vC0      =    -(vT * cos(Thet0) - vM * cos(Del0));        %Alg(4)
aMc0     =    N * vM * Omg0;                              %Alg(1)
if abs(aMc0 / 9.8) >= AccMaxG, aMc0 = AccMaxG * 9.8 * sign(aMc0); end

X0       =    [Rmt0; Lam0; Omg0; GamT0; GamM0; aM];
Alg0     =    [aMc0; Del0; Thet0; vC0];
tk       =    0;
bf       =    1;                    % before flyby flag
rr       =    0;                    % negative range rate flag
kk       =    0;
n        =    length(X0);
```

Guidance Cycle

```
%  Simulation Cycle
%
Rmt      =    Rmt0;
hwait    =    waitbar(0,'Please wait...');

while bf

      waitbar((Rmt0 - Rmt) / Rmt0, hwait)

      kk                =    kk + 1;

      tspan(1, kk)      =    tk;
      Xspan(:, kk)      =    Xk;
      Algspan(:, kk)    =    Algk;

      %
      [~, Xspank]       =    ode45(@dX_PPN_NL_1ord, [tk, tk + Ts], Xk,
      OptODE);

      Xk       =    Xspank(end, :)';

      % X    = [1-Rmt,2-Lam,3-Omg,4-GamT,5-GamM, 6-aM]
      Rmt      =    Xk(1);
      Lam      =    Xk(2);
      Omg      =    Xk(3);
      GamT     =    Xk(4);
      GamM     =    Xk(5);
      aM       =    Xk(6);

      % Alg = [1-aMc,2-Del,3-Thet, 4-vC]
      Del      =    GamM - Lam;
      Thet     =    GamT - Lam;
      vC       =    -(vT * cos(Thet) - vM * cos(Del));
      aMc      =    N * vM * Omg;
      if abs(aMc / 9.8) >= AccMaxG, aMc = AccMaxG * 9.8 * sign(aMc); end
      Algk     =    [aMc, Del, Thet, vC];

      % range rate for miss distance
      Rdot     =    (abs(Rmt) - abs(Xspan(1, kk))) / Ts;
      if rr == 0
            if rdot < 0
                  rr = 1;     % first time of negative range rate
            end
      else
            if rdot >= 0
                  bf   =    0;   % non-negative range rate after period of
                  negative range rate
            end
      end
      %
      tk   =    tk + Ts;
end  %while
```

Derivatives Calculation

```
%%%
 function    dx  =  dX_PPN_NL_1ord(~, x)
%Calculation of State Vector Derivatives
%  for Nonlinear Relative Motion Model
%  with First Order Dynamics
%-------------------------------------------------------------
```

```
% Model State Vector
%   X    = [1-Rmt, 2-Lam, 3-Omg, 4-GamT, 5-GamM, 6-aM]
% Algebraic Variables
%   Alg = [1-aMc, 2-Del, 3-Thet, 4-vC]
%--------------------------------------------------------------
%
%  X    = [1-Rmt,2-Lam,3-Omg,4-GamT,5-GamM,6-aM]

      dx       =   zeros(n, 1);

      rho      =   x(1);
      lambda   =   x(2);
      omega    =   x(3);
      gammaT   =   x(4);
      gammaM   =   x(5);
      am       =   x(6);
      %

      %  Alg = [1-aMc,2-Del,3-Thet, 4-vC]
      aMc      =   Algk(1);
      Del      =   Algk(2);
      Thet     =   Algk(3);
      vC       =   Algk(4);


      dx(1)    =   vT * cos(Thet) - vM * cos(Del);
      dx(2)    =   omega;

      x1_s     =   max(abs(rho), 1000) * sign(rho);
      dx(3)    =   -2 * (vT * cos(Thet) - vM * cos(Del)) * omega / x1_s ...
                   + aT * cos(Thet) / x1_s - am * cos(Del) / x1_s;

      dx(4)    =   aT / vT;
      dx(5)    =   am / vM;
      dx(6)    =   (-am + aMc) / tauM;

end
```

Miss Distance Calculation

```
tspan    =   tspan(1, 1 : kk);
t_fin    =   tspan(1, kk);
Xspan    =   Xspan(:, 1 : kk);
Algspan =   Algspan(:, 1 : kk);

% miss distance
miss   =   Xspan(1, end) * sin(Xspan(2, end));
fprintf('Mis dist.=%6.5g (m)   t_fin=%6.5g (sec)\n', miss, t_fin)
```

תקציר

בעבודה זאת אנו לומדים את בעיית היציבות של מערכות הנחייה בשיטת ניווט יחסי (Proportional Navigation, PN) בעזרת מספר כלים אנליטיים ונומריים במטרה לחקור בצורה מלאה את חוג ההנחיה, מפקודת התאוצה ועד מרחק ההחטאה.

ניווט יחסי היא השיטה הנפוצה ביותר להנחיית טילים. לרוב, חוק ההנחיה מבוטא כך: $a_c = N \cdot v_m \cdot \dot{\lambda}$, כאשר a_c היא פקודת תאוצה, N הוא הגבר, בדרך כלל נלקח בין הערכים 3-5, ו- v_m היא מהירות הטיל. λ נקראת זווית קו הראיה, זווית זו נוצרת מהמפגש של הקו הישיר בין הטיל למטרה (קו הראייה) וקו ייחוס שרירותי. כיוון שחוק ההנחיה פרופורציונאלי לנגזרת זווית קו הראייה מובן שהחוק מנסה לבטל את קצב קו הראייה, כלומר לאפס את הנגזרת ולשמור על זווית קו ראייה קבועה. אופן הפעולה כולל מדידת קצב זווית קו הראיה, חישוב פקודת תאוצה על פי חוק ההנחיה וביצוע הפקודה ע״י מערכת בקרה שמסיטה משטחי היגוי בהתאם לפקודה וגורמת לתנועה בכיוון הרצוי.

מערכת ההנחיה מתוארת על ידי מודל לא לינארי שכולל את המשוואות הקינמטיות, את חוק ההנחיה ואת הדינמיקה של הטיל. המודל הלא לינארי המלא הוא הבסיס לכל ניתוח שיבוצע.

לינאריזציה של מודל מערכת ההנחיה מבוצעת סביב מצב ייחוס אידיאלי שבו הטיל והמטרה נמצאים במסלול התנגשות ובנקודת עבודה זו לא נדרשות מצד הטיל המיירט פקודות הנחיה כדי לתקן שגיאות הכוון. על בסיס המודל הלינארי המקורב מבוצע ניתוח תיאורטי שלא ניתן לקיים באמצעות המשוואות המדויקות. המודל שמתקבל כתוצאה מהלינאריזציה הינו תלוי בזמן (LTV) באופן שהמקדמים של מטריצות המצב בזמן נכפלים בזמן הספירה לאחור – t_{go} (time-to-go).

תיאוריה שיכולה ללמוד את המודל הלינארי היא השיטה שמרחיבה את תיאורית ליאפונוב למערכות שתלויות בזמן סופי. כיוון שמקדמי המודל הלינארי תלויים בזמן הספירה לאחור t_{go} קיימת נקודה סינגולרית בסוף התהליך, כלומר ברגע החליפה של הטיל והמטרה. לסוג כזה של משוואות תיאורית ליאפונוב אינה ישימה והשיטה המוצעת מתמודדת עם הבעיה על ידי חקירת ההתנהגות של מספר הולך וגדל של זמן הטיסה. כתוצאה מכך מתקבלים תנאים מספיקים ליציבות ויציבות אסימפטוטית. על בסיס השיטה פותחה יכולת לתכנון פרמטרי של המרכיבים בחוג ההנחיה.

בהמשך מבוצע ניתוח לתת-המערכת שכוללת בתוכה את המשוואה שמתארת את הטווח בין הטיל למטרה בכל רגע. מוצא המערכת מתאר את מרחק ההחטאה (miss distance) ומכאן חשיבות חלק זה של העבודה, שמבוצע באופן נומרי. האתגר בחלק זה הוא ההתמודדות עם שני אינטגרטורים שמופיעים במוצא תת המערכת הקודמת, שתלויה בזמן הספירה לאחור $\mathrm{t_{go}}$ ושנחקרה בשיטת ליאפונוב מורחבת. החקירה כאן תתבצע ביחס ליציבות במובן כניסה-מוצא, שהיא תגובת המערכת למצב התחלתי. התוצאות המתקבלות הינם תנאים מספיקים ליציבות ויציבות אסימפטוטית.

המשך העבודה מסודר כך: בפרק ראשון רקע מתמטי ופיזיקלי הנדרש למערכת ההנחיה. פרק שני מוקדש ללינאריזציה של המערכת. בפרק שלישי ניתוח תת-המערכת שמרכיביה תלויים בזמן t_{go} ובפרק רביעי חקירה של התנהגות מרחק ההחטאה. פרק חמישי – סיכום ומסקנות. בנספח 1 סימולציה למערכת הנחיה בשיטת ניווט יחסי.

אוניברסיטת אריאל בשומרון

הפקולטה להנדסת חשמל ואלקטרוניקה

חקירות יציבות בשיטת ליאפונוב מורחבת ובסימולציות למערכות ניווט יחסי

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